Finite Automata

Part Three

From Last Time

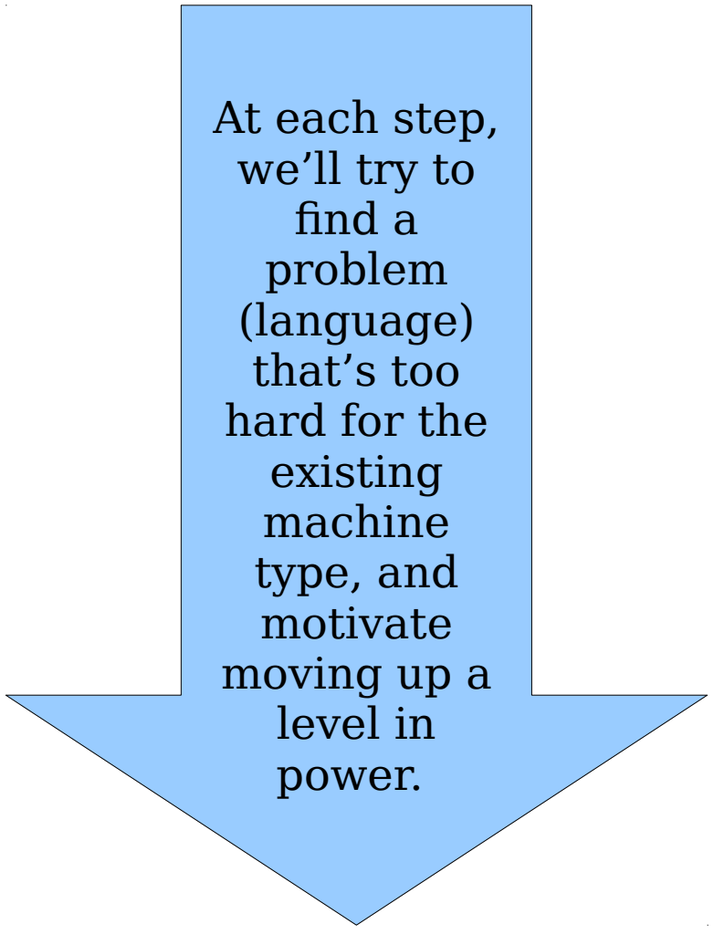
NFAs and DFAs

- **DFAs** (Deterministic Finite Automata)
  - are machines for accepting/rejecting strings.
  - The language of a DFA is the set of strings it accepts.
  - The set of languages for which there exists a DFA is called the **Regular Languages**.
    - *Spoiler: not all languages are Regular! We'll find some that you can't build a DFA for, due to DFA's finite limitation.*
- **NFAs** (Nondeterministic Finite Automata)
  - are DFAs but with some bonus superpowers of having more options for how we move from state to state.

Quick orientation to material for the rest of the quarter

Automata Progression

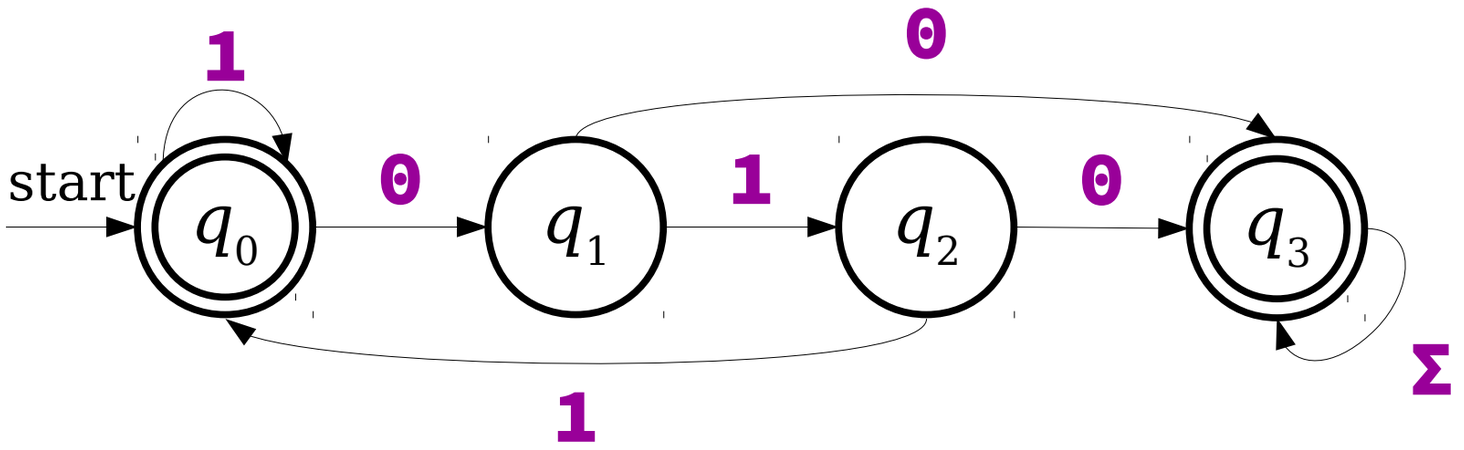
- **DFAs** (Deterministic Finite Automata)
- **NFAs** (DFA + superpowers of guessing)
- **PDAs** (NFA + a stack to use as memory)
  We'll actually explore an equivalent mechanism called CFGs, you'll see soon…
- **Turing Machines** (DFA or NFA + an infinite array to use as memory)



New Stuff!

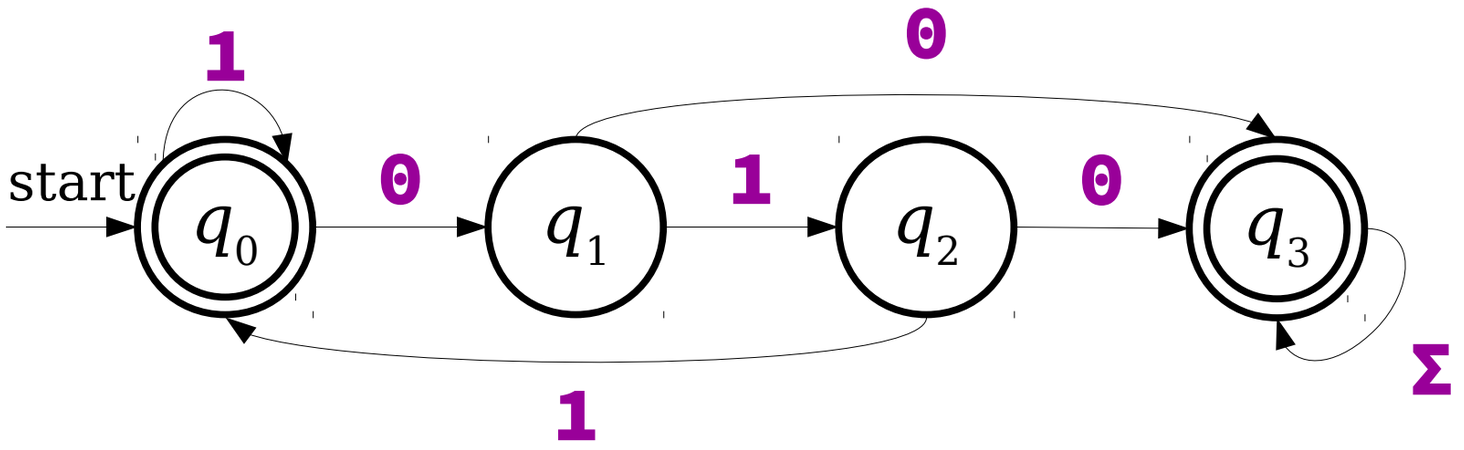
Table Representation of DFAs

Tabular DFAs



	0	1
q_0		
q_1		
q_2		
q_3		

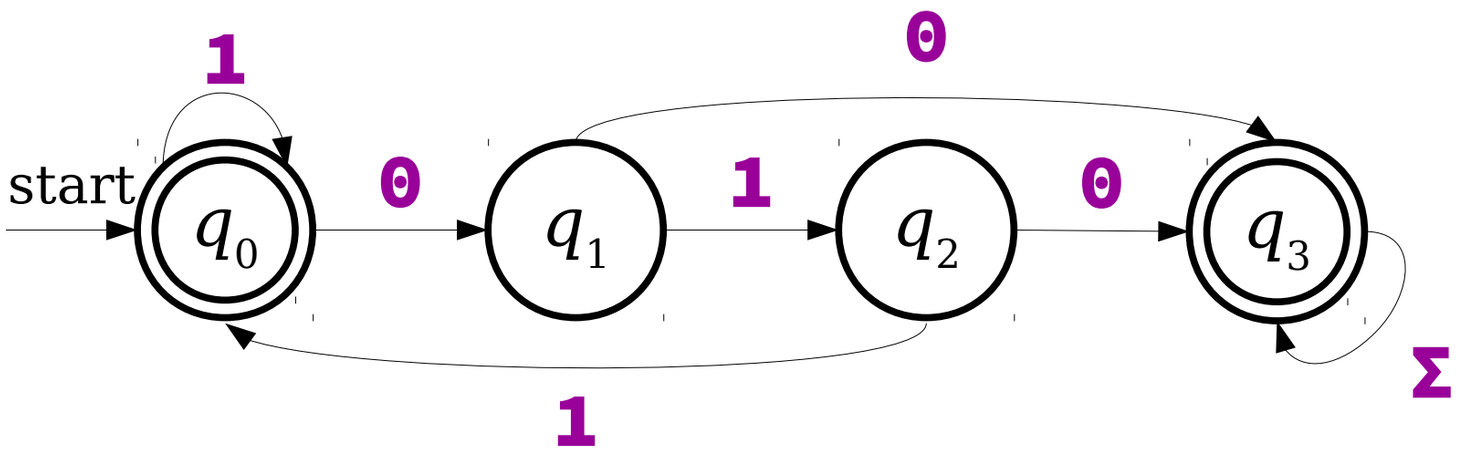
Tabular DFAs



	0	1
q_0	q_1	q_0
q_1		
q_2		
q_3		

Since this is the first row, it's the start state.

Tabular DFAs



Question:
What goes in the q_1 row of the table?

	0	1
q_0	q_1	q_0
q_1		
q_2		
q_3		

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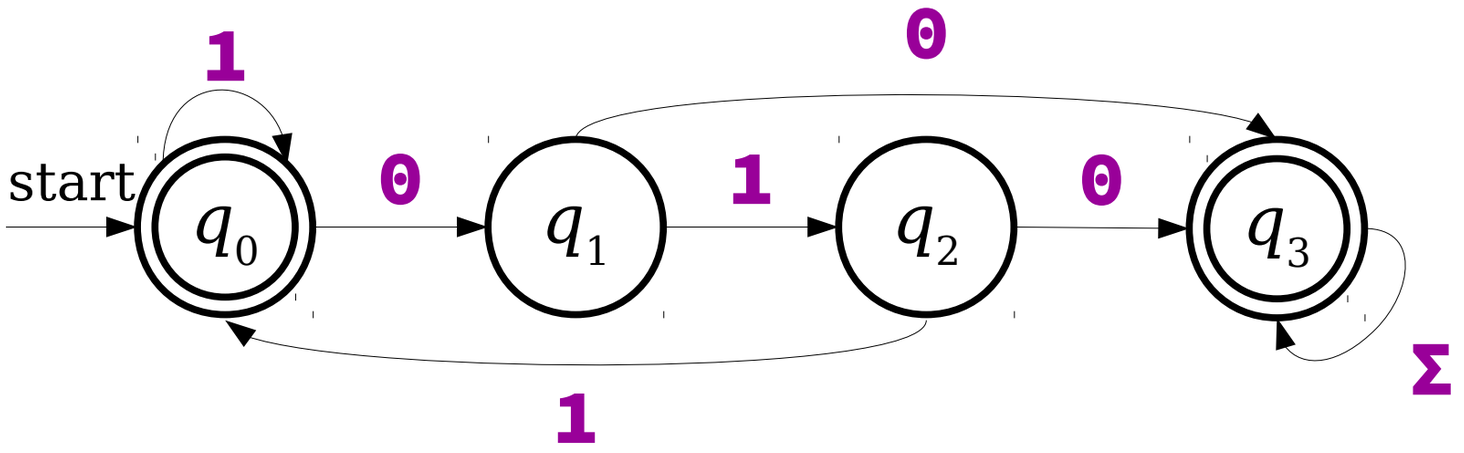
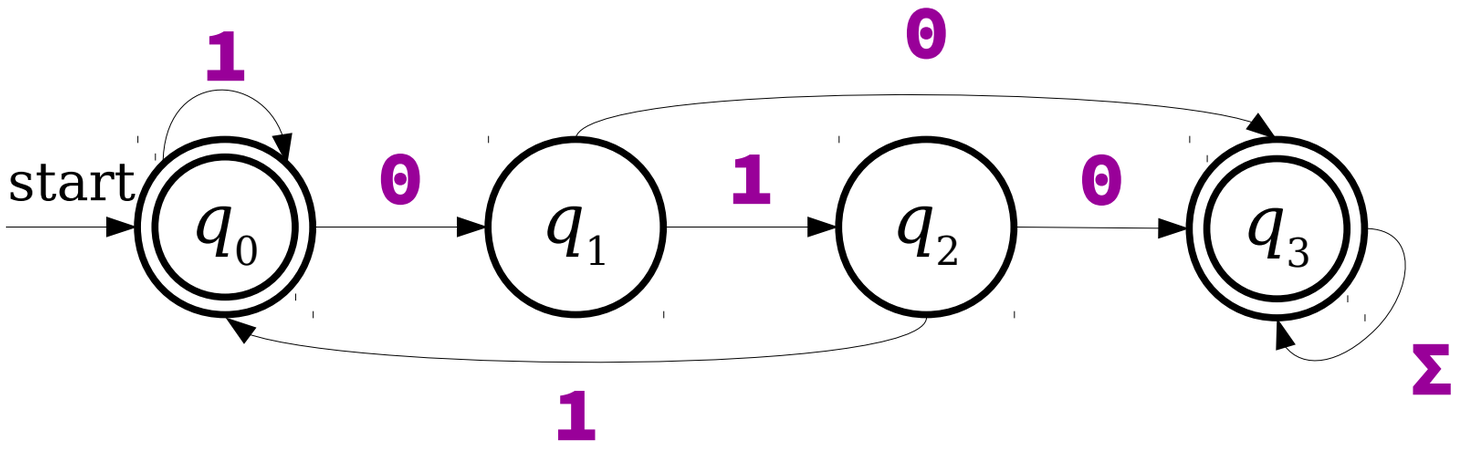
Tabular DFAs



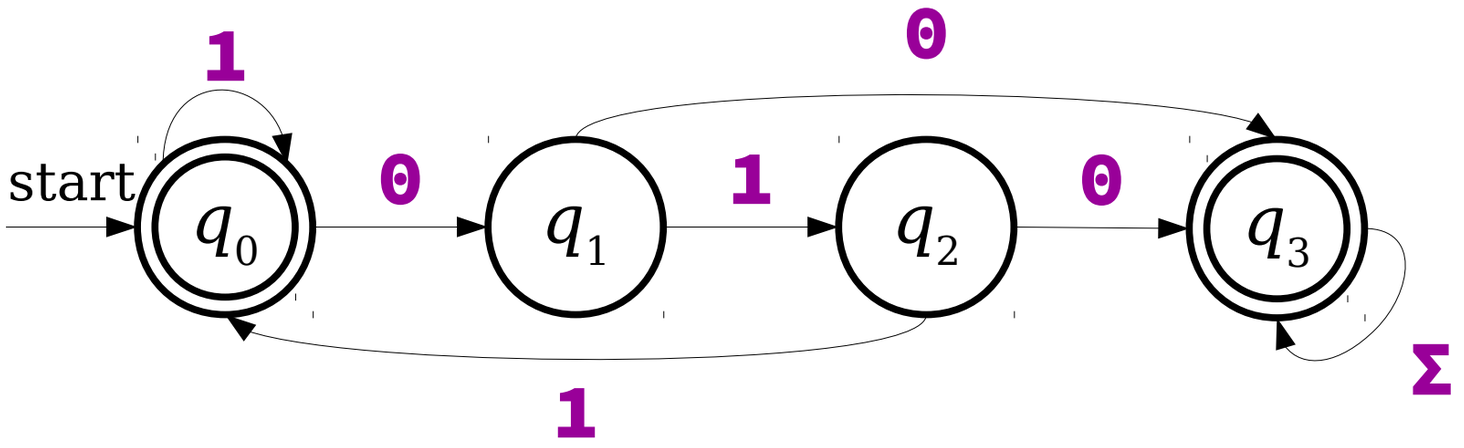
	0	1
q_0	q_1	q_0
q_1	q_3	q_2
q_2		
q_3		

Tabular DFAs



	0	1
q_0	q_1	q_0
q_1	q_3	q_2
q_2	q_3	q_0
q_3		

Tabular DFAs



(A)

	0	1	Σ
q_0	q_1	q_0	-
q_1	q_3	q_2	-
q_2	q_3	q_0	-
q_3	-	-	q_3

(B)

	0	1
q_0	q_1	q_0
q_1	q_3	q_2
q_2	q_3	q_0
q_3	q_3	q_3

Question:
How should we complete the table?

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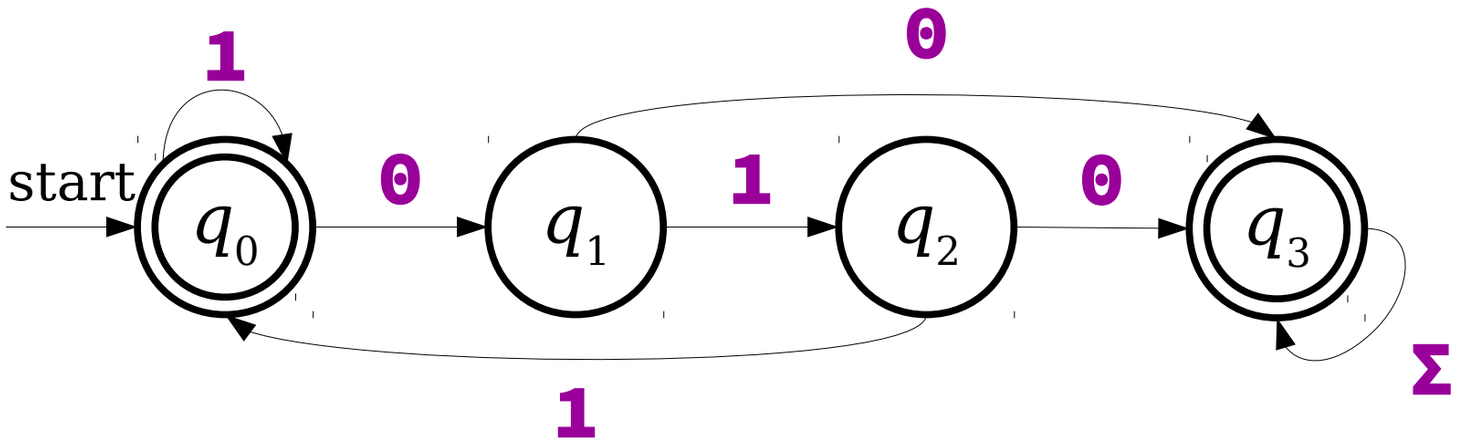
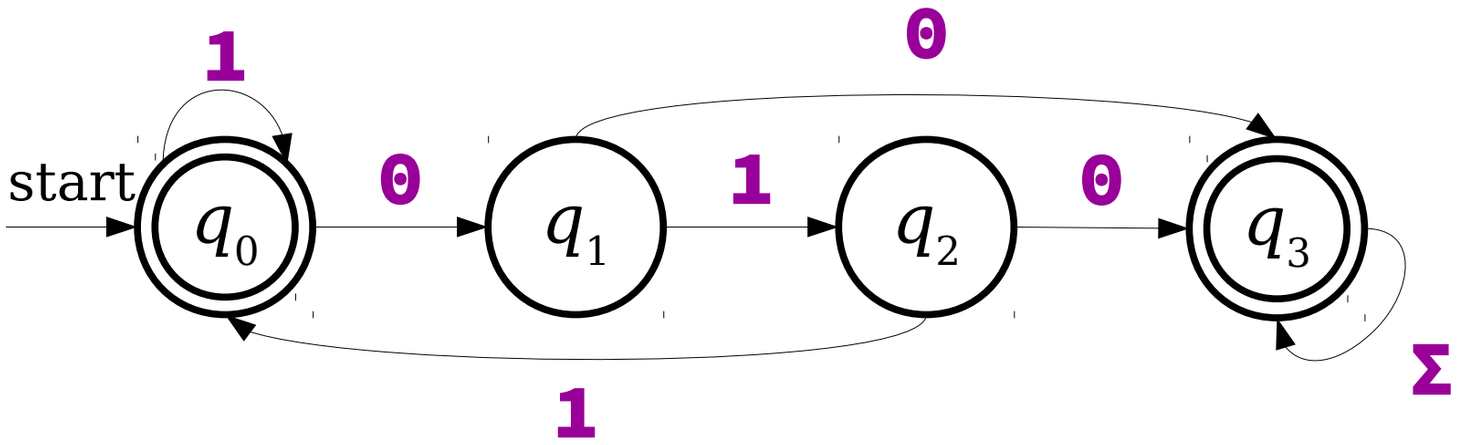
Tabular DFAs



	0	1
q_0	q_1	q_0
q_1	q_3	q_2
q_2	q_3	q_0
q_3	q_3	q_3

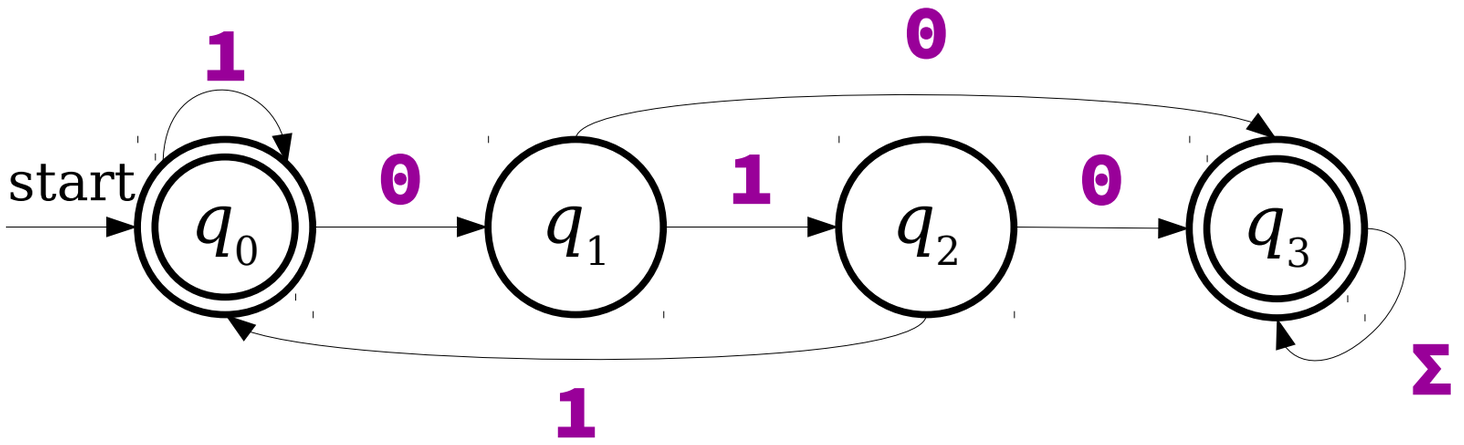
Tabular DFAs



	0	1
$*q_0$	q_1	q_0
q_1	q_3	q_2
q_2	q_3	q_0
$*q_3$	q_3	q_3

These stars indicate accepting states.

Tabular DFAs



	0	1
$*q_0$	q_1	q_0
q_1	q_3	q_2
q_2	q_3	q_0
$*q_3$	q_3	q_3

DFAs and NFAs

- Every language for which there exists a DFA also has an NFA.
- Why?
  - NFAs are just DFAs with (optional to use) extra superpowers
  - So every DFA is essentially already an NFA!

NFAs and DFAs

- What about the reverse?
- ***Question:*** Can any language recognized by an NFA also be recognized by a DFA?

Send in your hot (or cold) take!

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NFAs and DFAs

- What about the reverse?
- ***Question:*** Can any language recognized by an NFA also be recognized by a DFA?
  - Seems unlikely: NFAs can do everything DFA can do, *plus* they have superpowers of perfect guessing the future!

NFAs and DFAs

- What about the reverse?
- ***Question:*** Can any language recognized by an NFA also be recognized by a DFA?
  - Seems unlikely: NFAs can do everything DFA can do, *plus* they have superpowers of perfect guessing the future!
- Surprisingly, the answer is ***yes***!

NFAs and DFAs

- ***Theorem:*** For all languages L, if L is recognized by an NFA, then there exists a DFA that also recognizes L.
- ***Proof plan:***
  - ***Assume:*** Pick an arbitrary language L, for which an NFA exists
  - ***Want to show:*** We want to show that there exists a DFA for L.
  - Describe how we would construct a DFA with the same language, in a generalizable way. *For the next few slides, we'll ponder how to approach that…*

NFAs and DFAs

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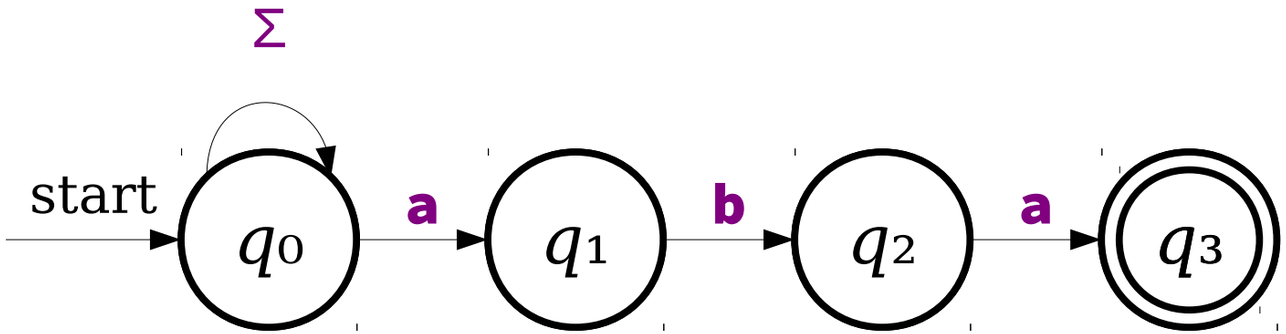
Key Insight:

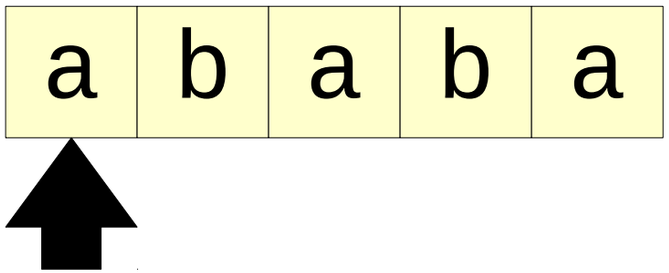
Our NFA transition function, and
Massive Parallelism
[from last time]

Massive Parallelism

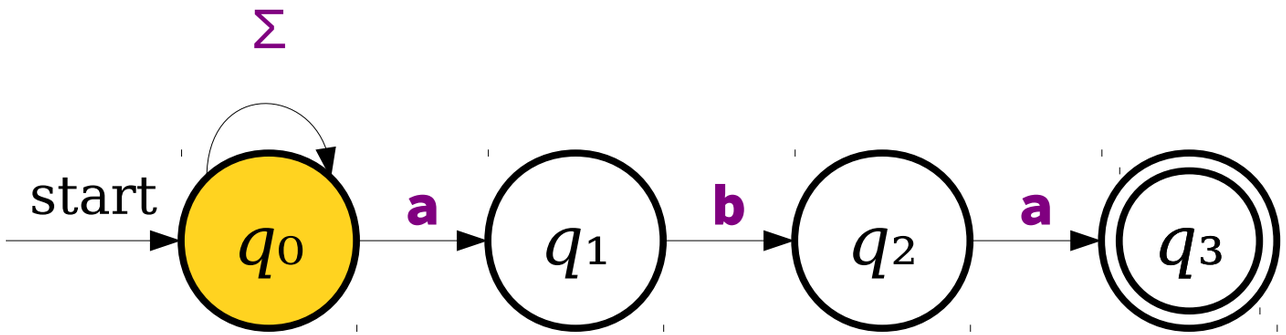
- An NFA can be thought of as a DFA that can be in many states at once.
- At each point in time, when the NFA needs to follow a transition, it tries all the options at the same time.

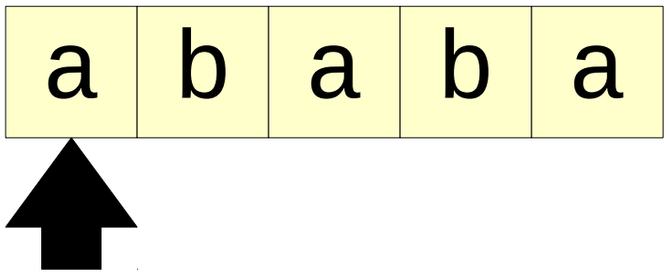
Massive Parallelism



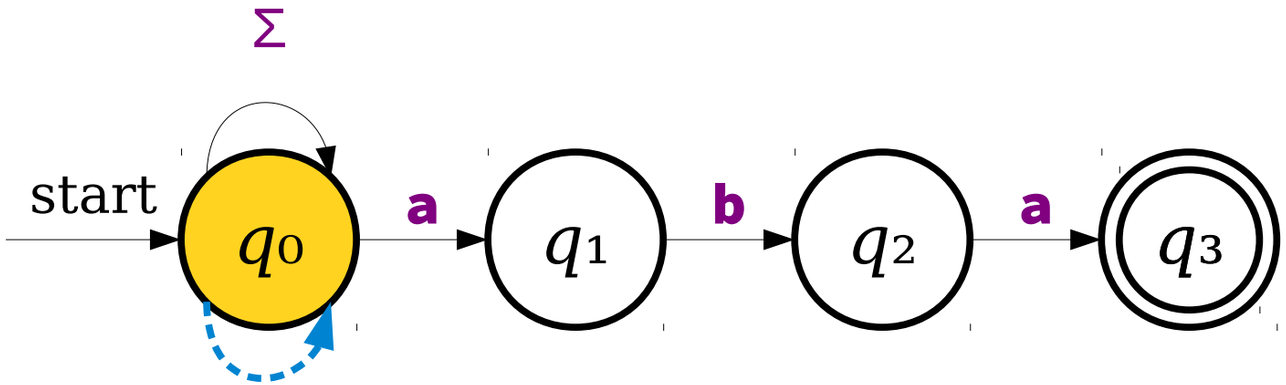


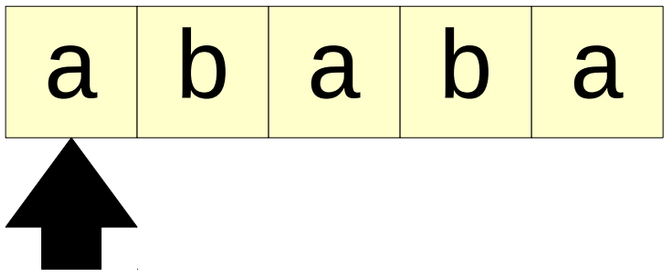
Massive Parallelism



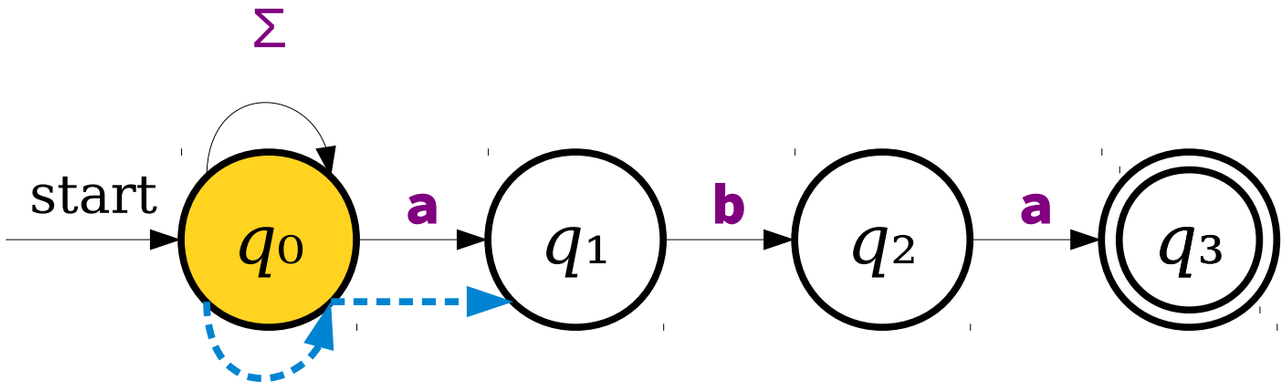


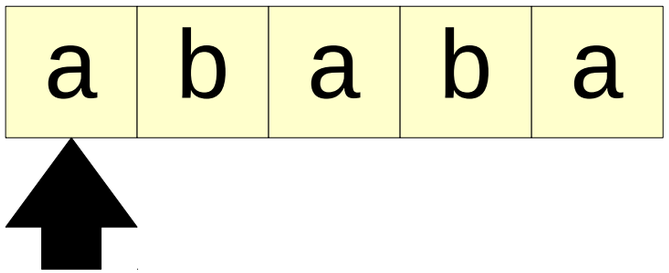
Massive Parallelism



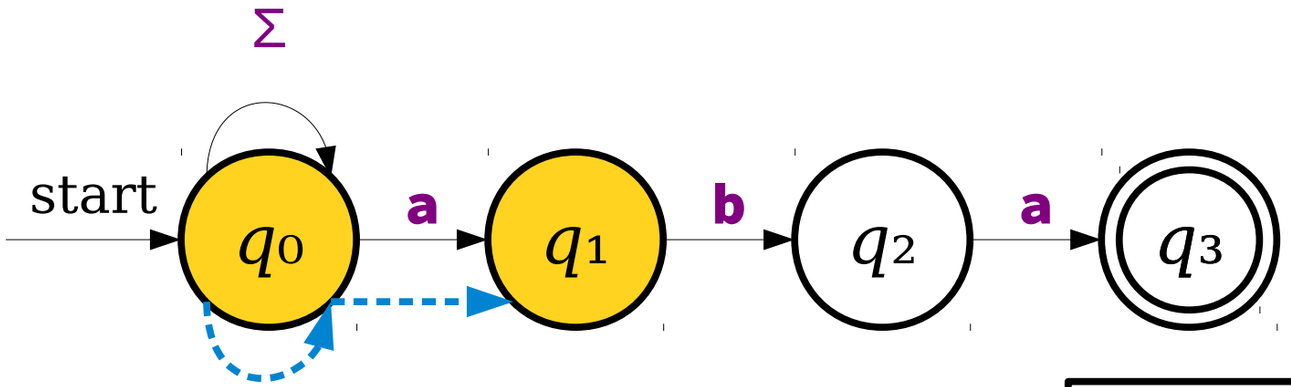


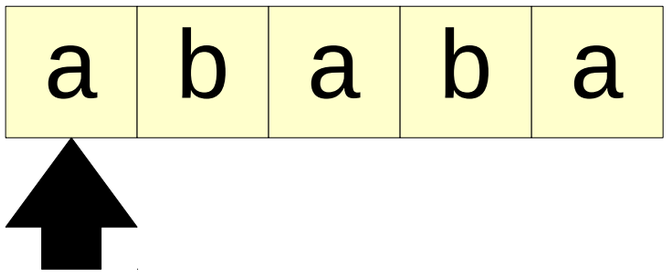
Massive Parallelism





Massive Parallelism

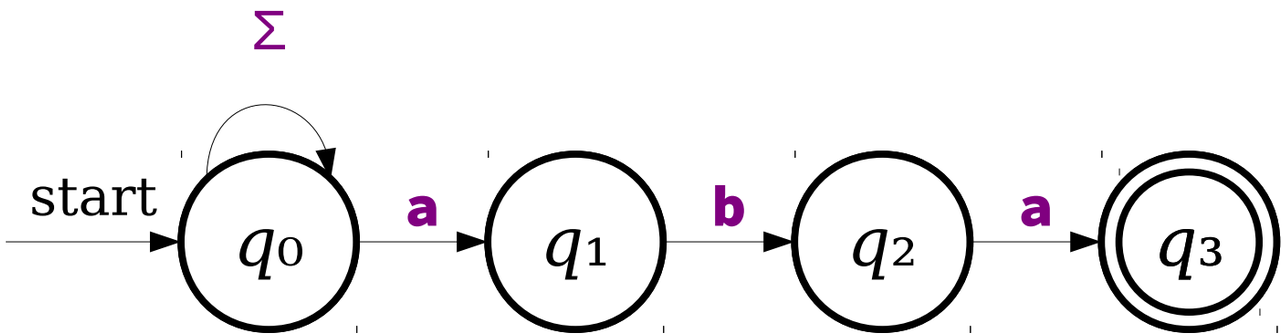




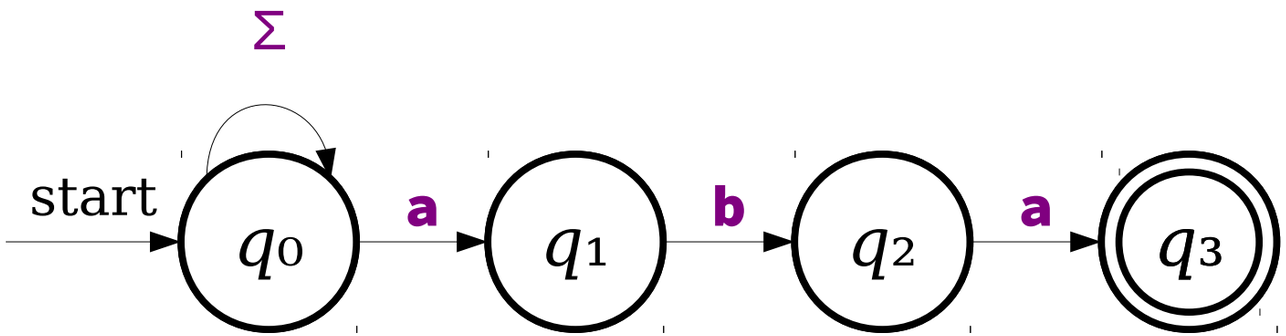
"Massive parallelism" matches how we defined the NFA transition function—we think of "where I am right now" as not just one state, but simultaneously all of some subset of the states.

NFA: $\delta : (\wp(S) \times \Sigma) \rightarrow \wp(S)$

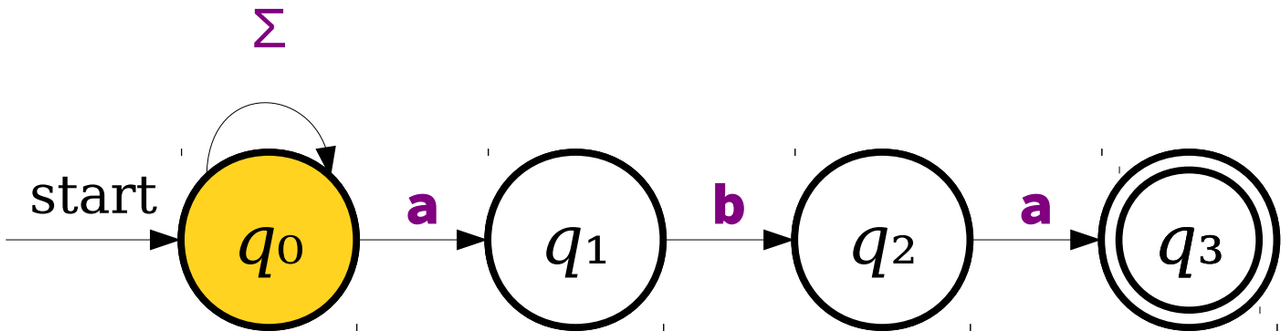
Putting our NFA insight
into the DFA table representation...



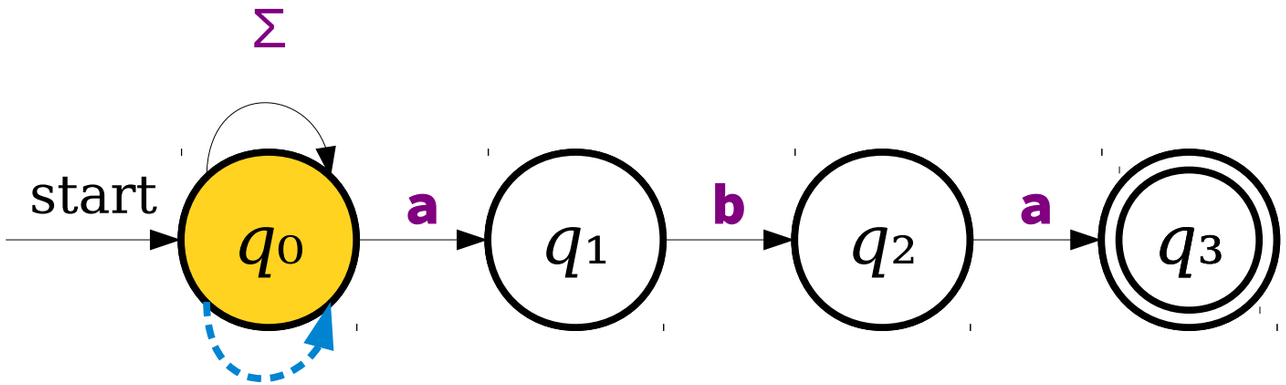
	a
$\{q_0\}$	$\{q_0, q_1\}$



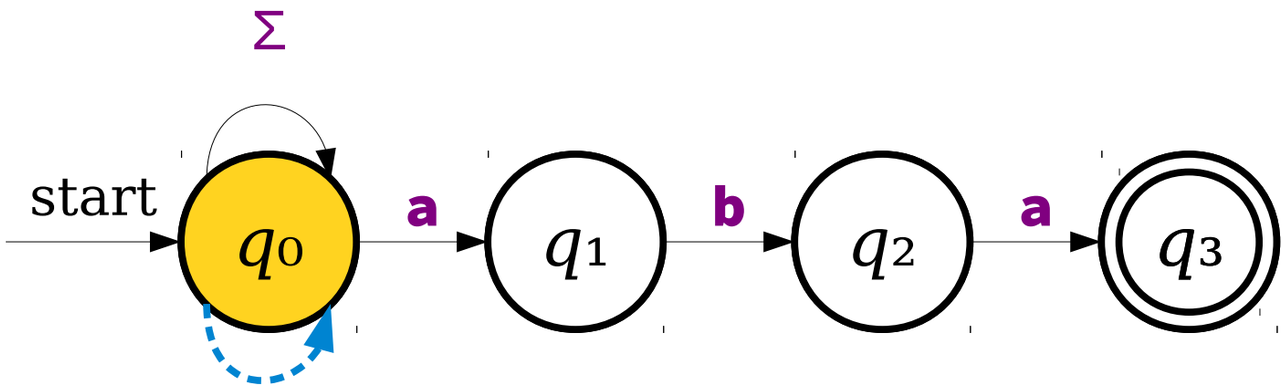
	a	b
$\{q_0\}$	$\{q_0, q_1\}$	



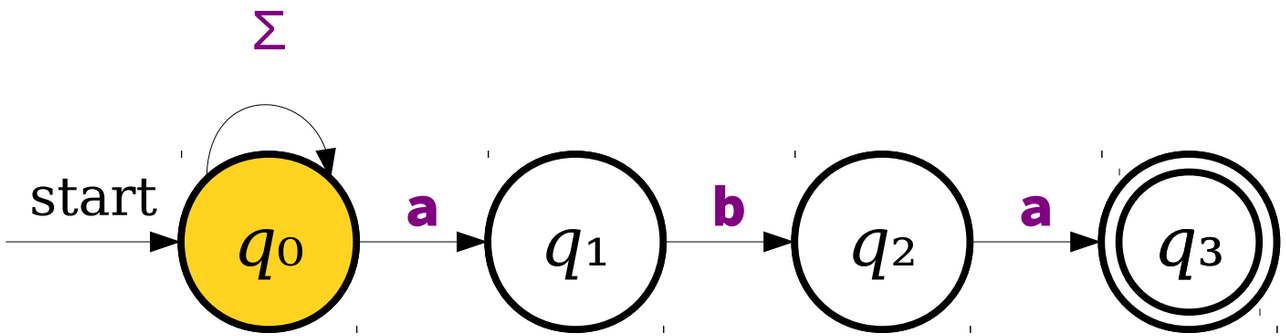
	a	b
$\{q_0\}$	$\{q_0, q_1\}$	



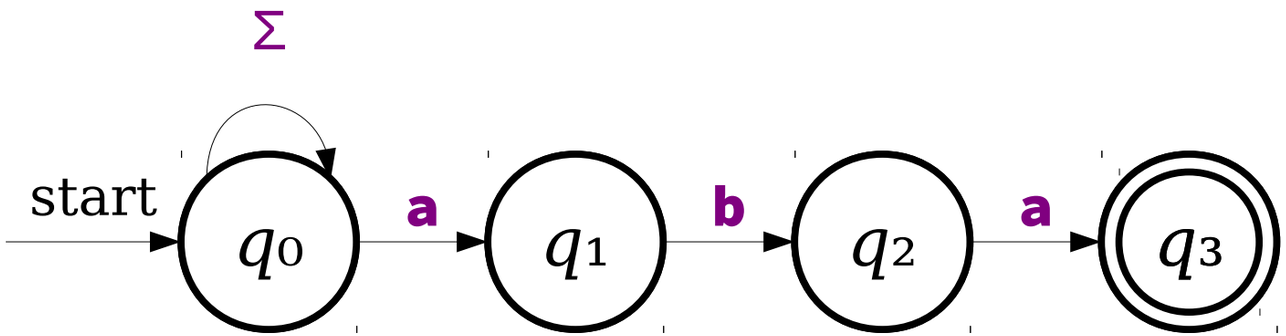
	a	b
$\{q_0\}$	$\{q_0, q_1\}$	



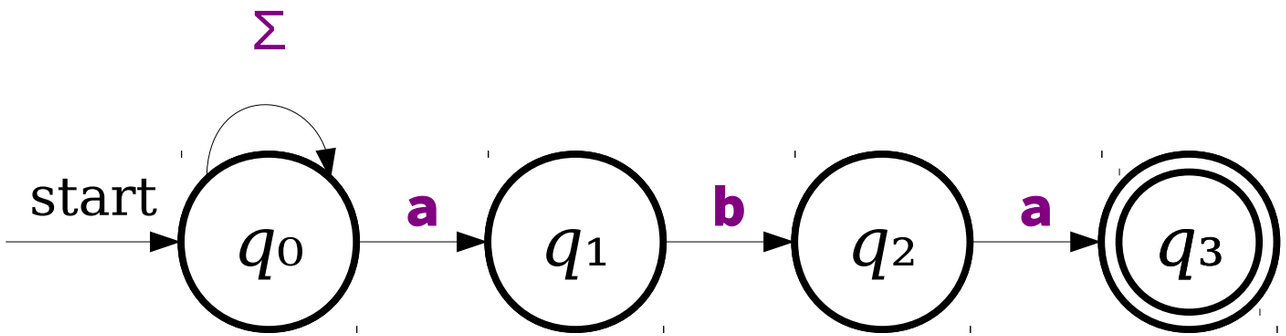
	a	b
$\{q_0\}$	$\{q_0, q_1\}$	$\{q_0\}$



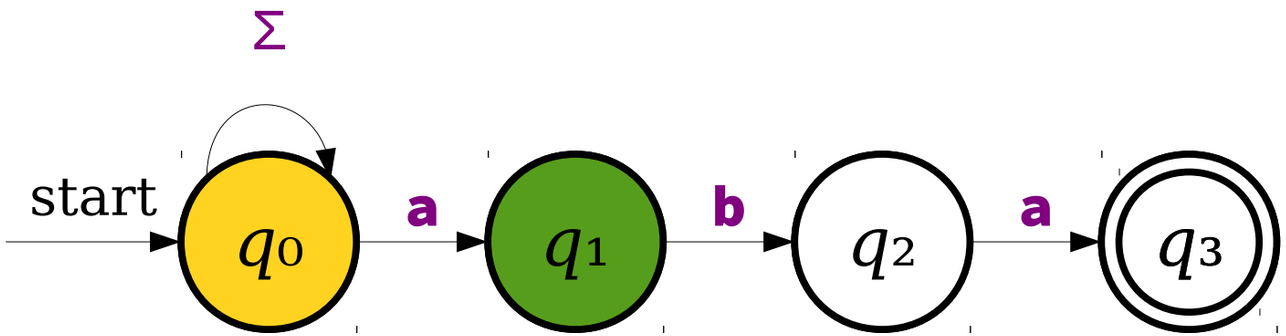
	a	b
$\{q_0\}$	$\{q_0, q_1\}$	$\{q_0\}$



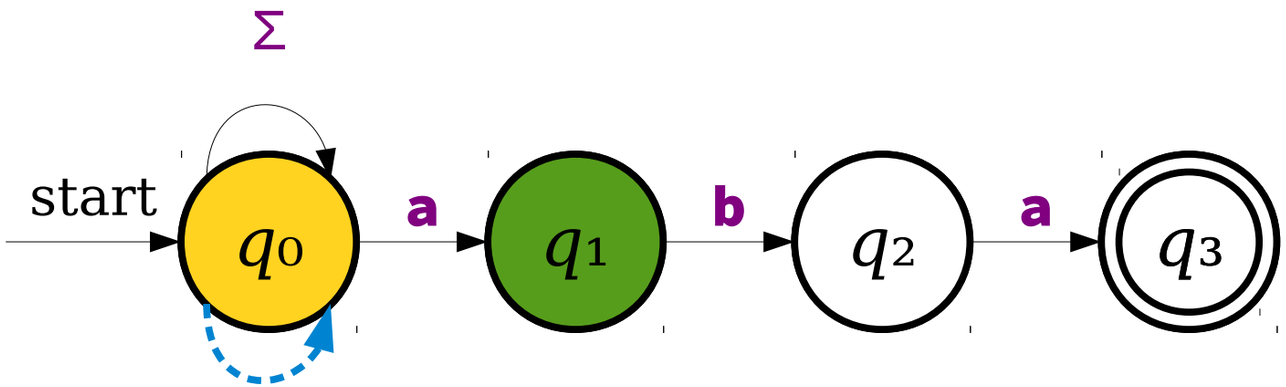
	a	b
$\{q_0\}$	$\{q_0, q_1\}$	$\{q_0\}$



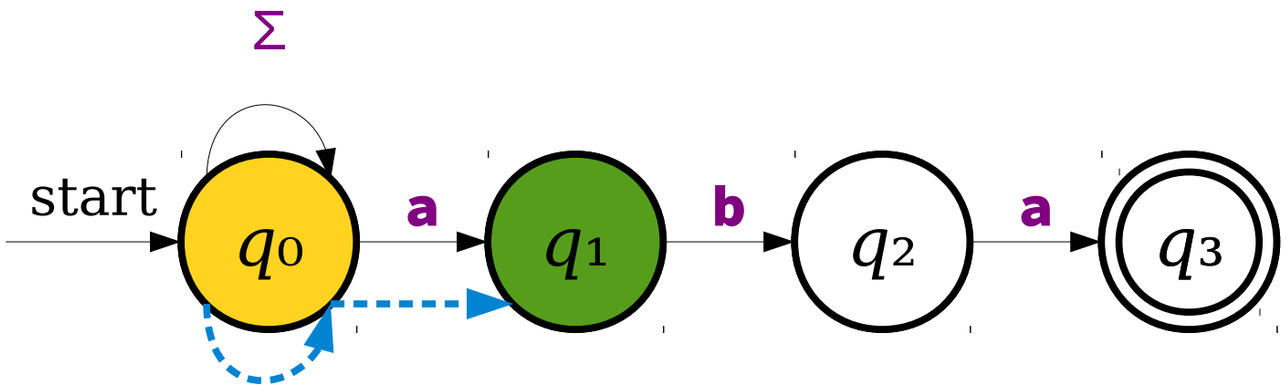
	a	b
$\{q_0\}$	$\{q_0, q_1\}$	$\{q_0\}$
$\{q_0, q_1\}$		



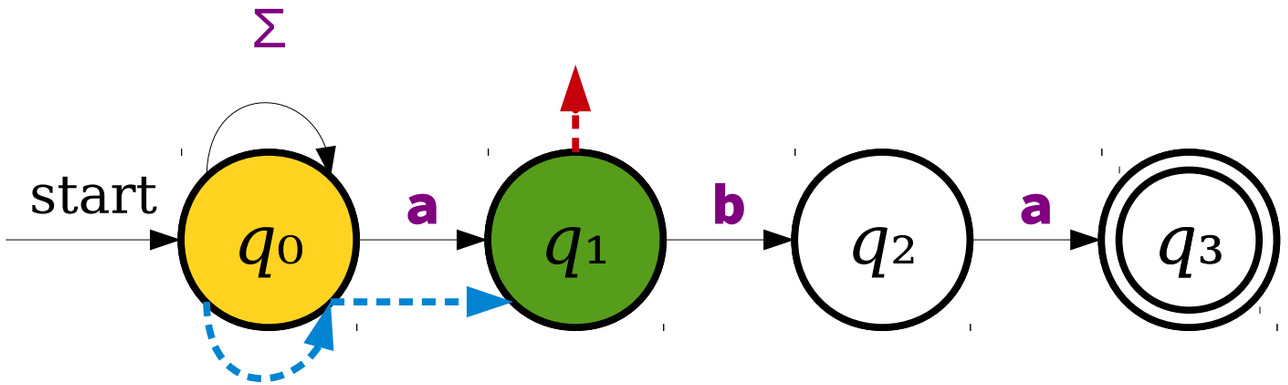
	a	b
$\{q_0\}$	$\{q_0, q_1\}$	$\{q_0\}$
$\{q_0, q_1\}$		



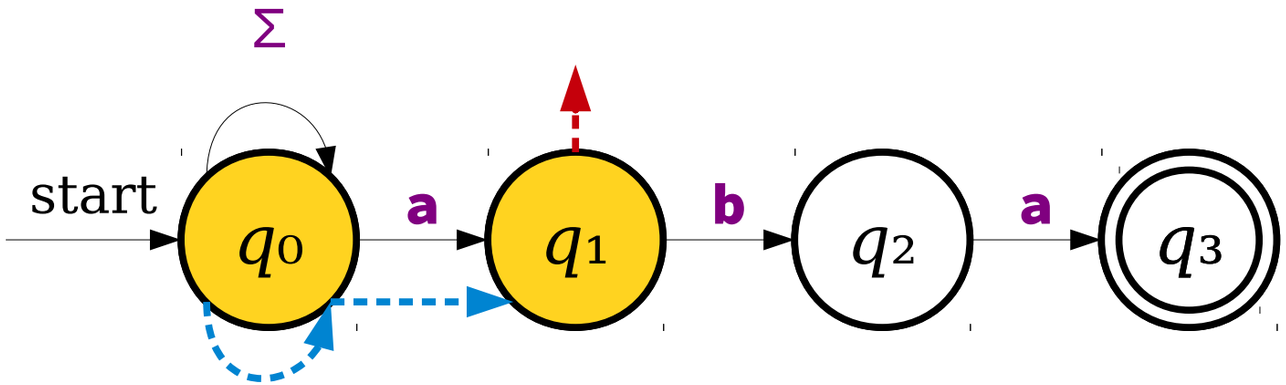
	a	b
$\{q_0\}$	$\{q_0, q_1\}$	$\{q_0\}$
$\{q_0, q_1\}$		



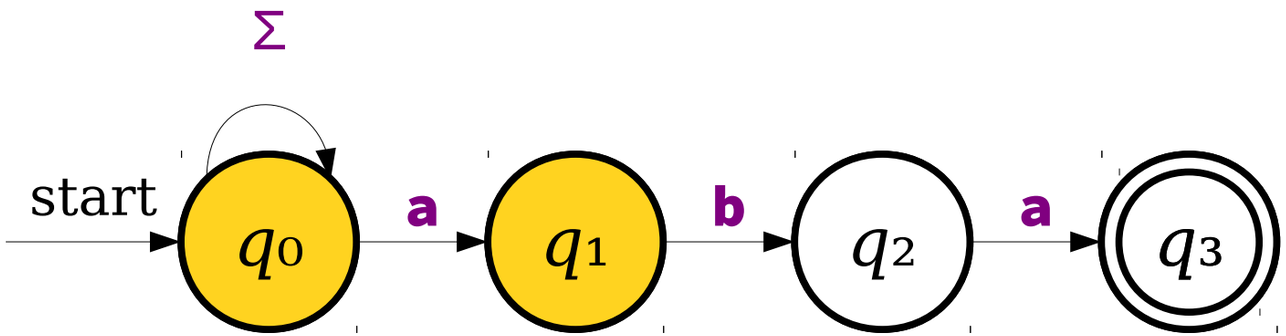
	a	b
$\{q_0\}$	$\{q_0, q_1\}$	$\{q_0\}$
$\{q_0, q_1\}$		



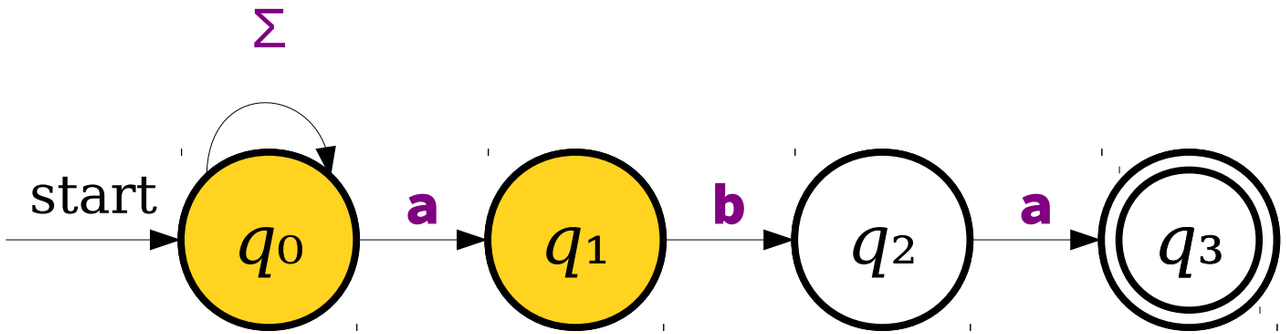
	a	b
$\{q_0\}$	$\{q_0, q_1\}$	$\{q_0\}$
$\{q_0, q_1\}$		



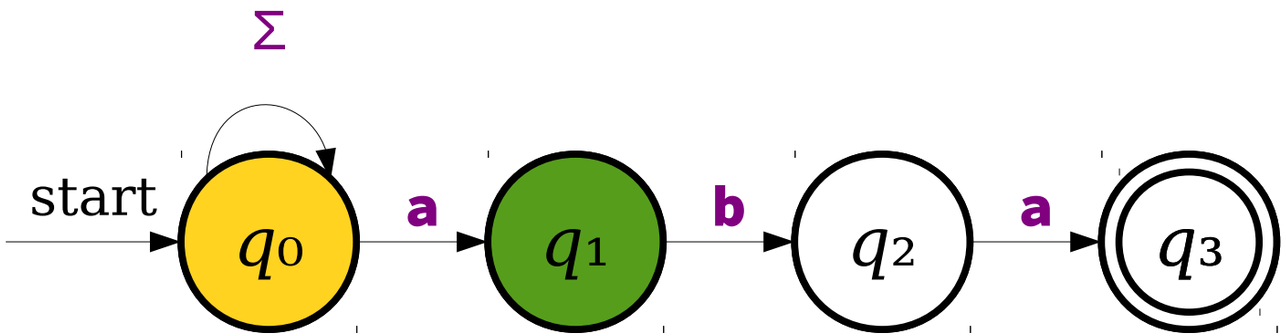
	a	b
$\{q_0\}$	$\{q_0, q_1\}$	$\{q_0\}$
$\{q_0, q_1\}$		



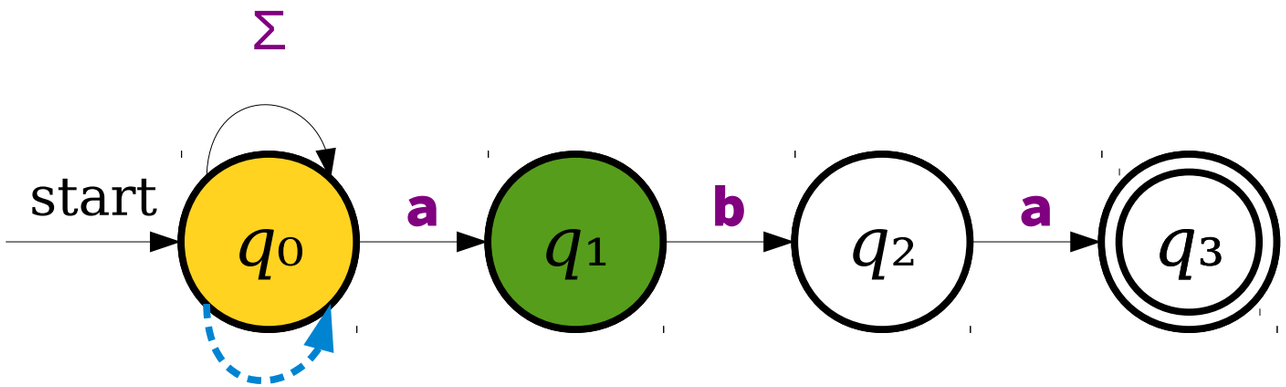
	a	b
$\{q_0\}$	$\{q_0, q_1\}$	$\{q_0\}$
$\{q_0, q_1\}$		



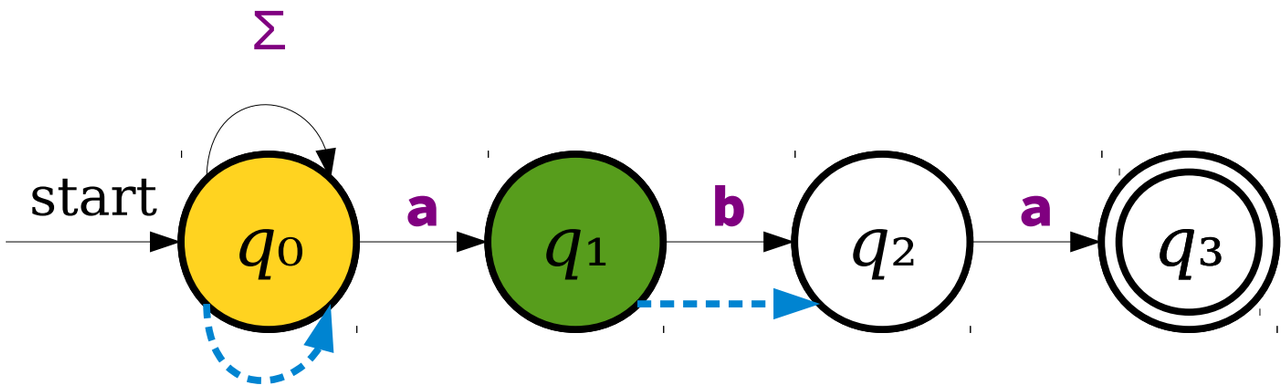
	a	b
$\{q_0\}$	$\{q_0, q_1\}$	$\{q_0\}$
$\{q_0, q_1\}$	$\{q_0, q_1\}$	



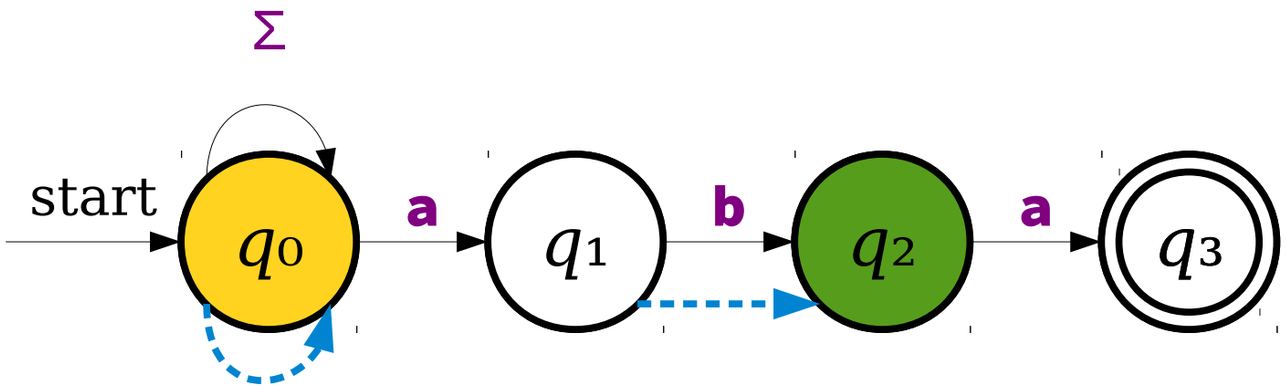
	a	b
$\{q_0\}$	$\{q_0, q_1\}$	$\{q_0\}$
$\{q_0, q_1\}$	$\{q_0, q_1\}$	



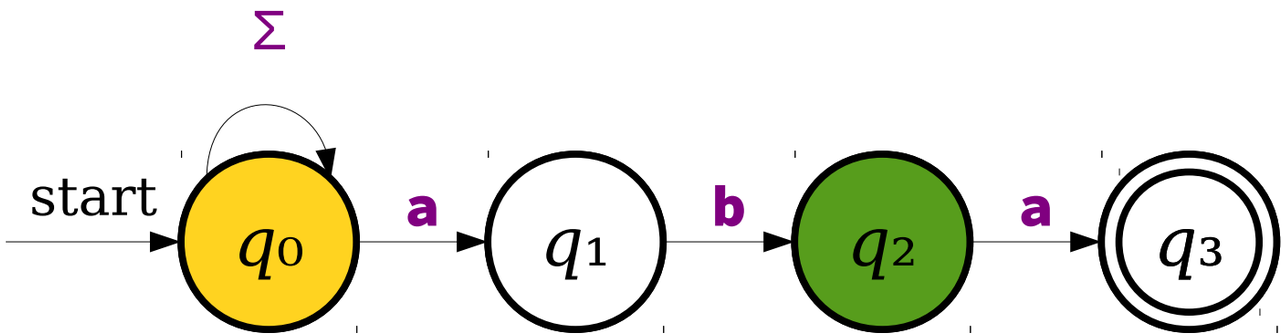
	a	b
$\{q_0\}$	$\{q_0, q_1\}$	$\{q_0\}$
$\{q_0, q_1\}$	$\{q_0, q_1\}$	



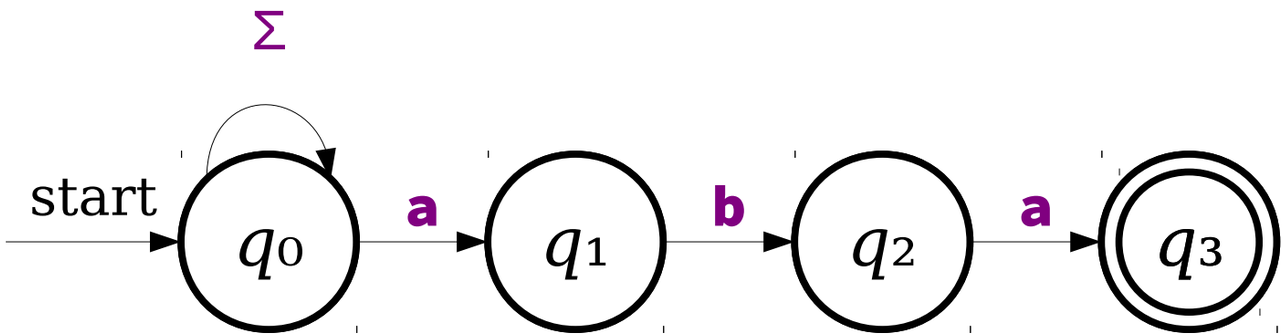
	a	b
$\{q_0\}$	$\{q_0, q_1\}$	$\{q_0\}$
$\{q_0, q_1\}$	$\{q_0, q_1\}$	



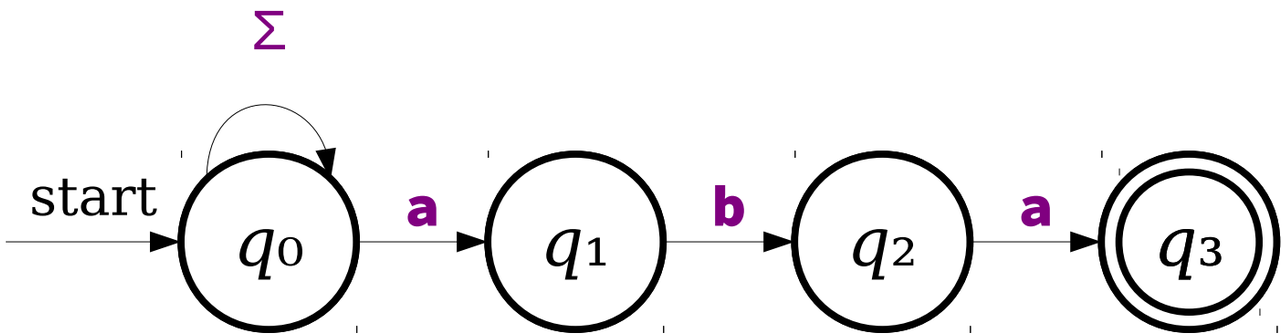
	a	b
$\{q_0\}$	$\{q_0, q_1\}$	$\{q_0\}$
$\{q_0, q_1\}$	$\{q_0, q_1\}$	



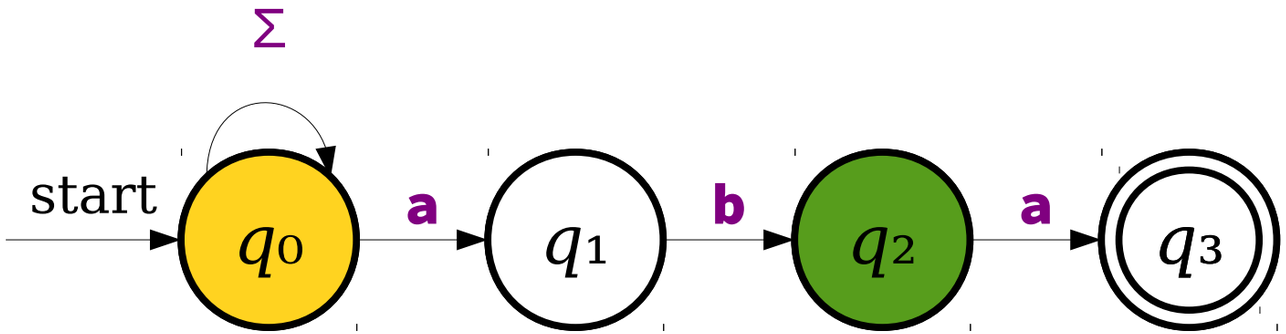
	a	b
$\{q_0\}$	$\{q_0, q_1\}$	$\{q_0\}$
$\{q_0, q_1\}$	$\{q_0, q_1\}$	$\{q_0, q_2\}$



	a	b
$\{q_0\}$	$\{q_0, q_1\}$	$\{q_0\}$
$\{q_0, q_1\}$	$\{q_0, q_1\}$	$\{q_0, q_2\}$



	a	b
$\{q_0\}$	$\{q_0, q_1\}$	$\{q_0\}$
$\{q_0, q_1\}$	$\{q_0, q_1\}$	$\{q_0, q_2\}$
$\{q_0, q_2\}$		



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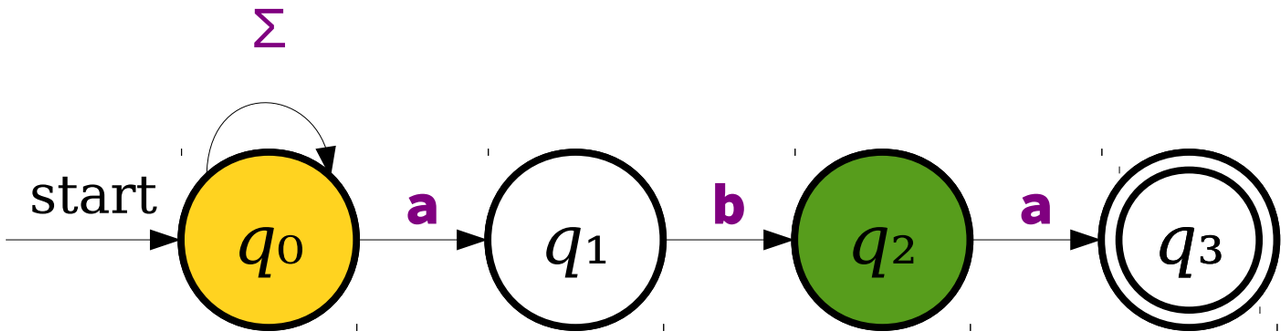

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$\{q_0, q_2\}$		

Quick check:
What are the contents of the next row? Answer like
“a = {..}, b = {...}”
all in one response.

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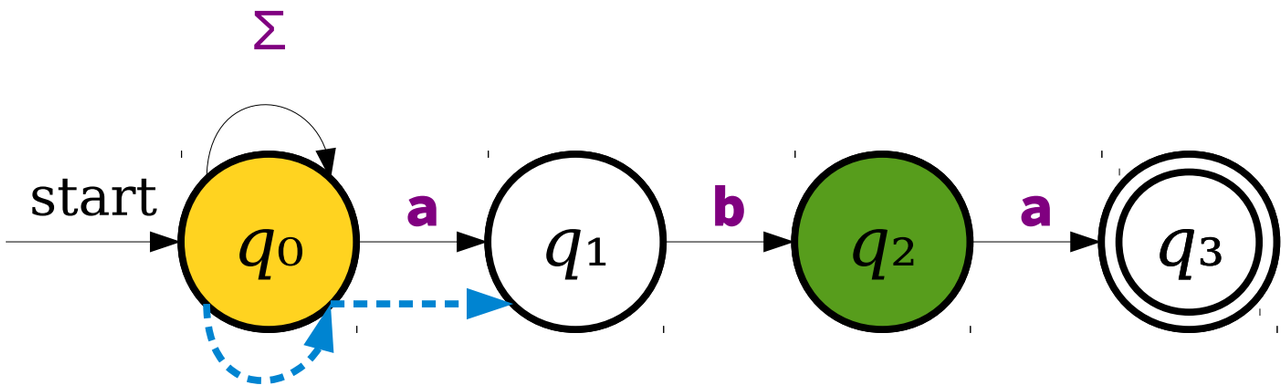

	a	b
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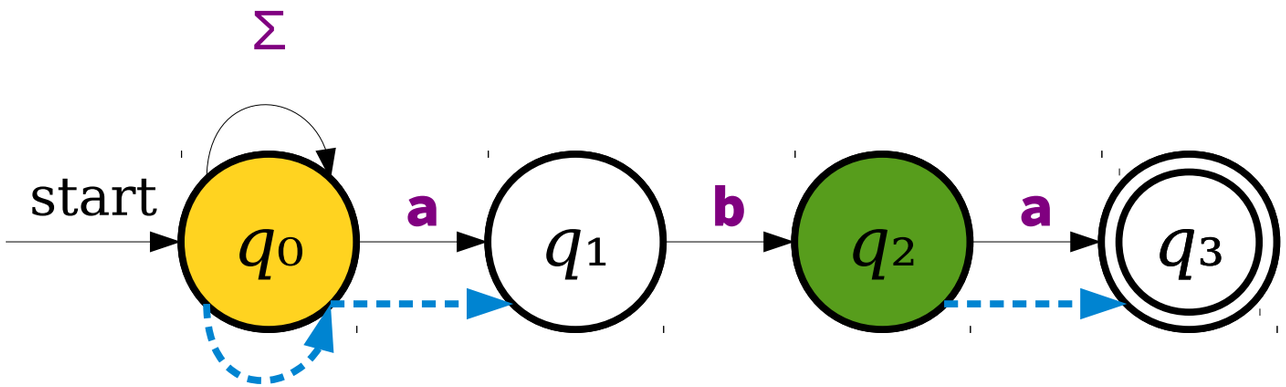

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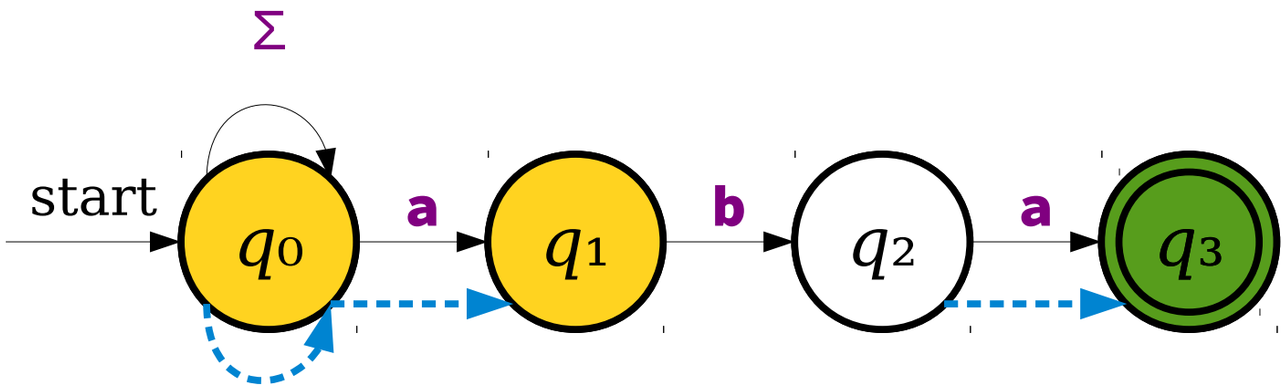

	a	b
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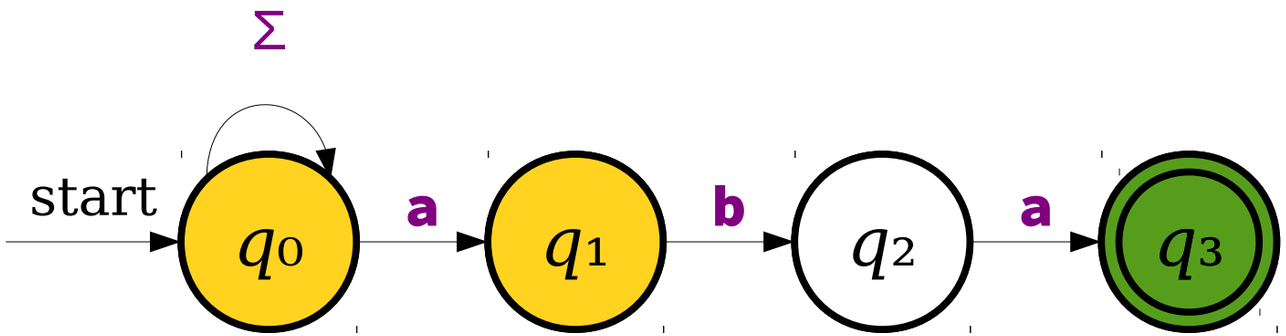

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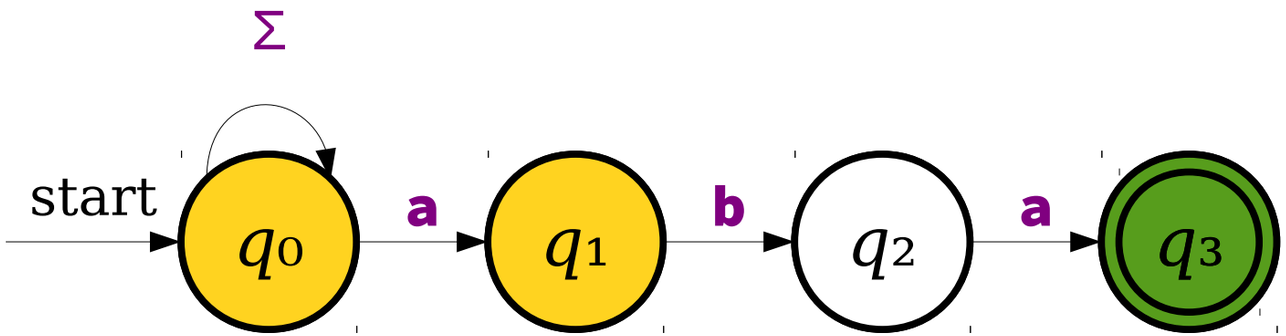

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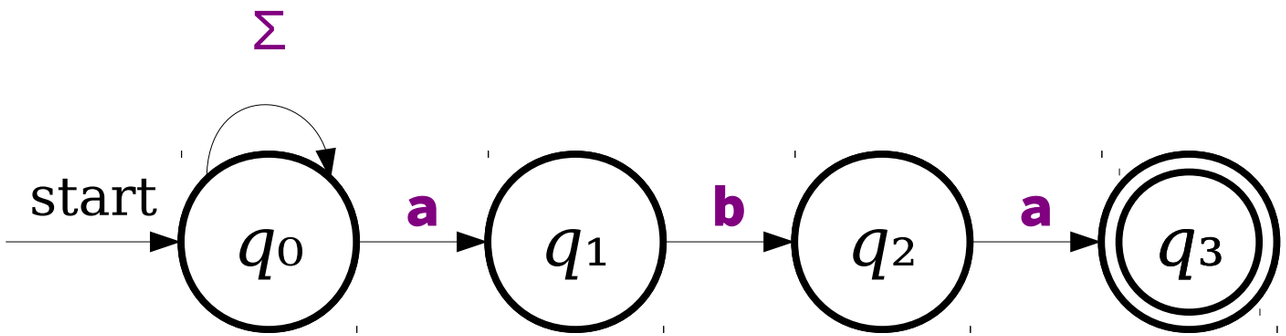

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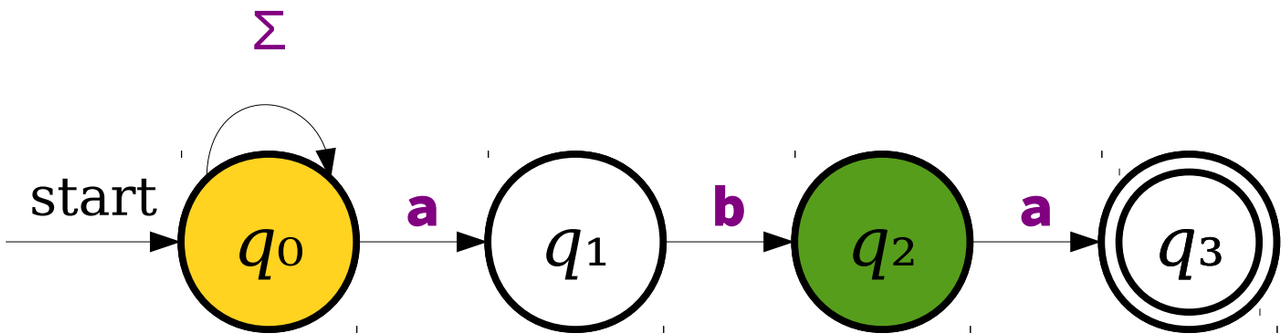

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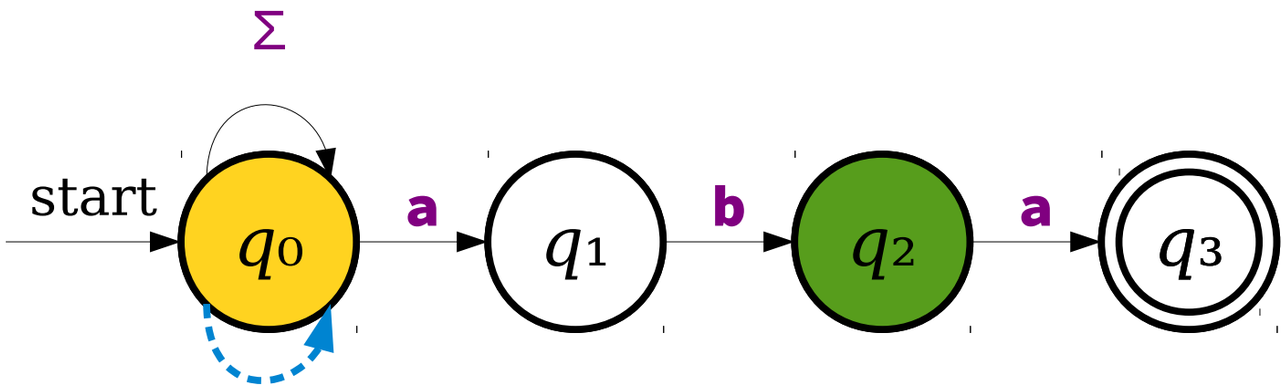

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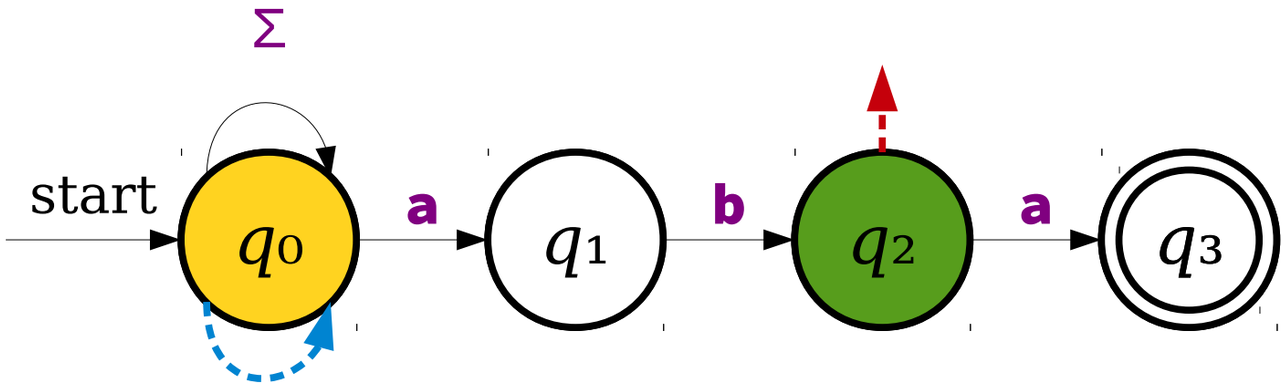

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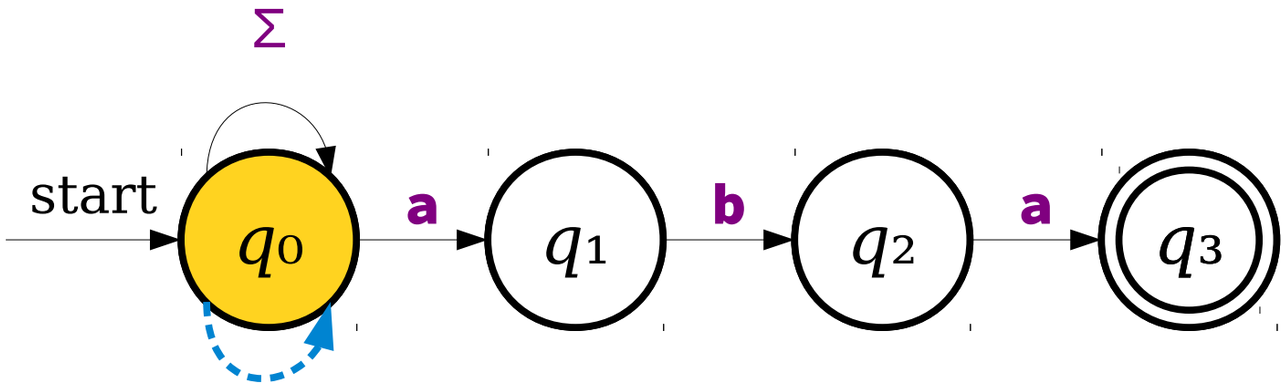

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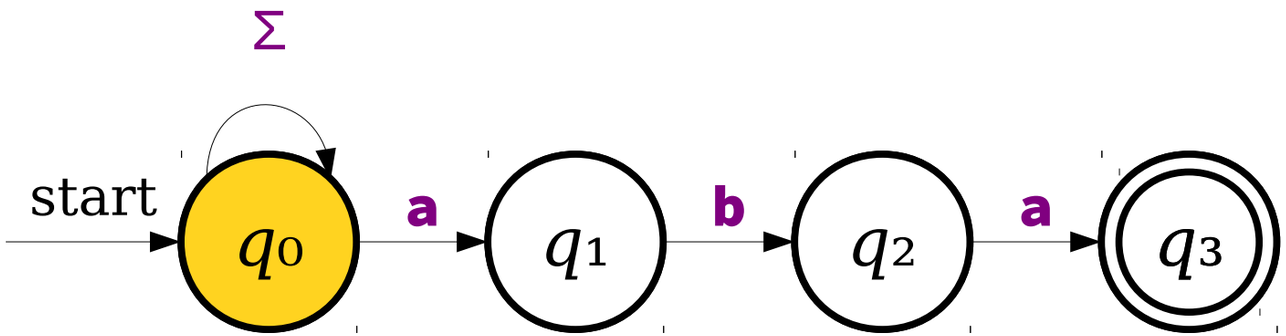

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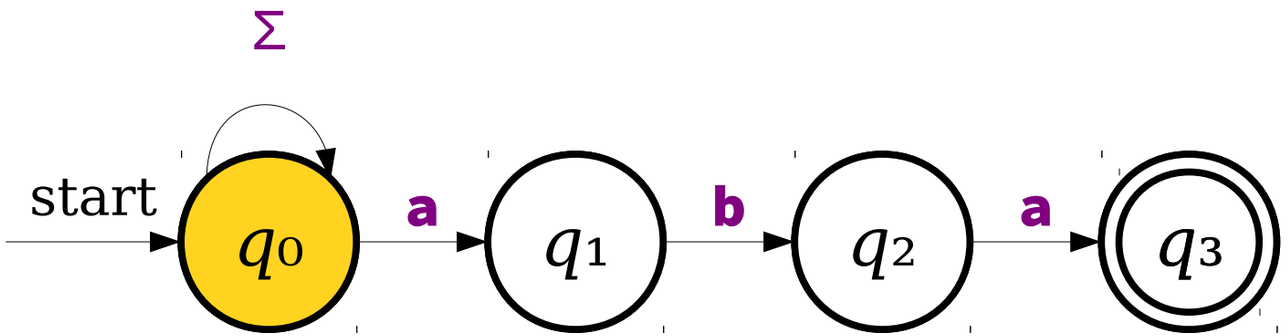

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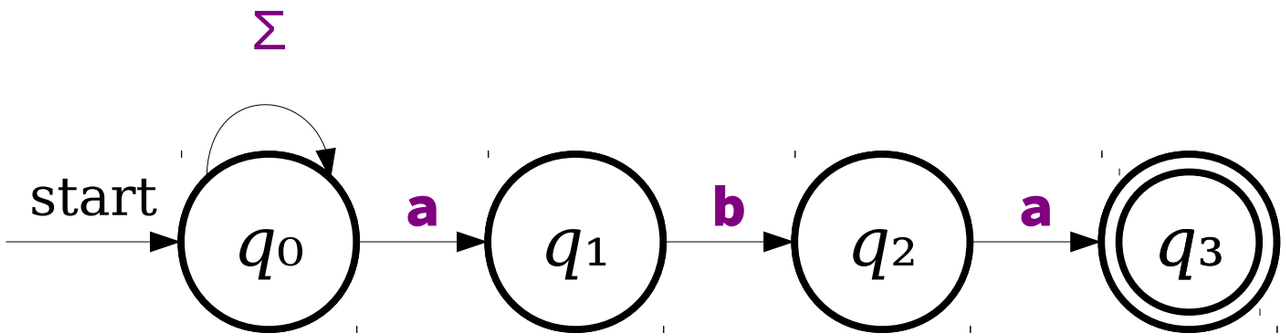

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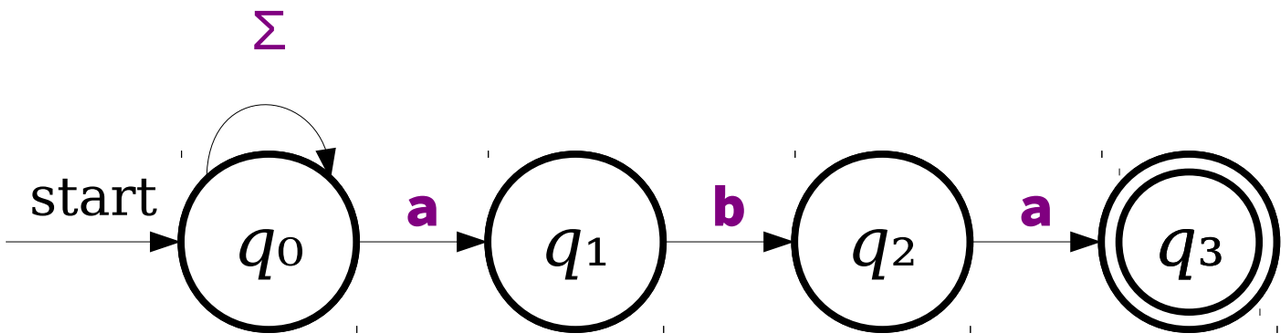

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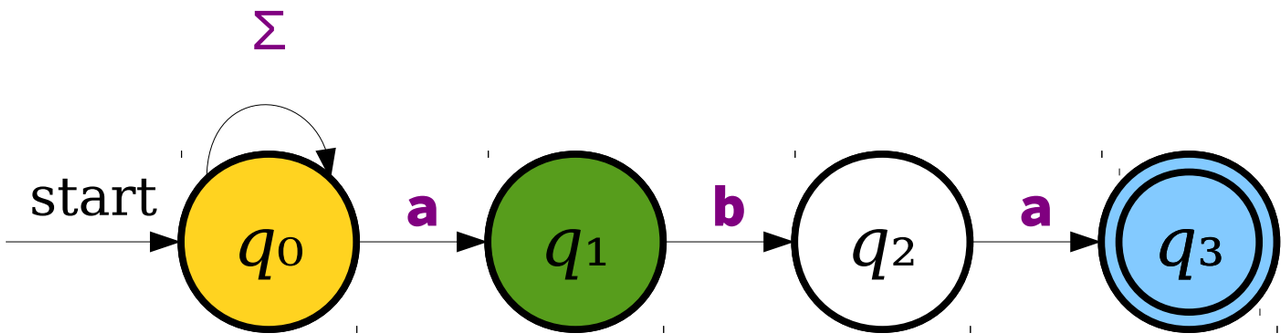

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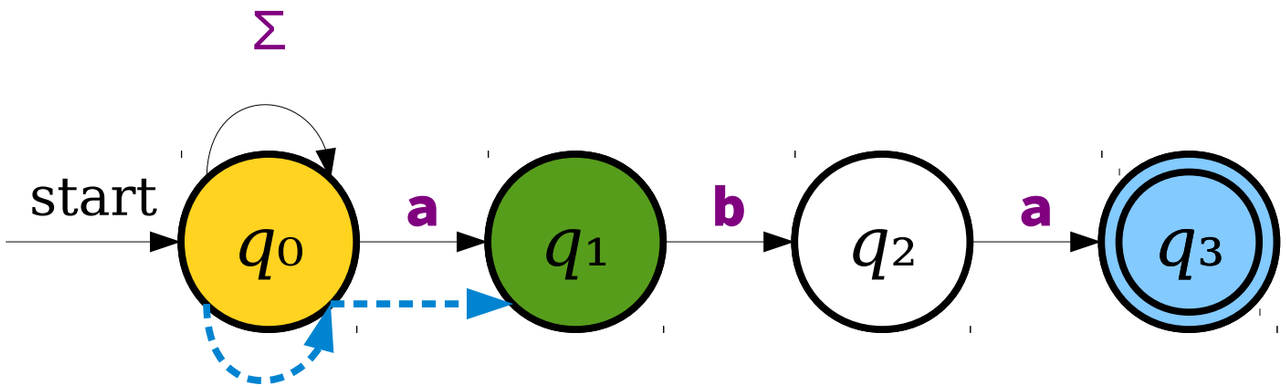

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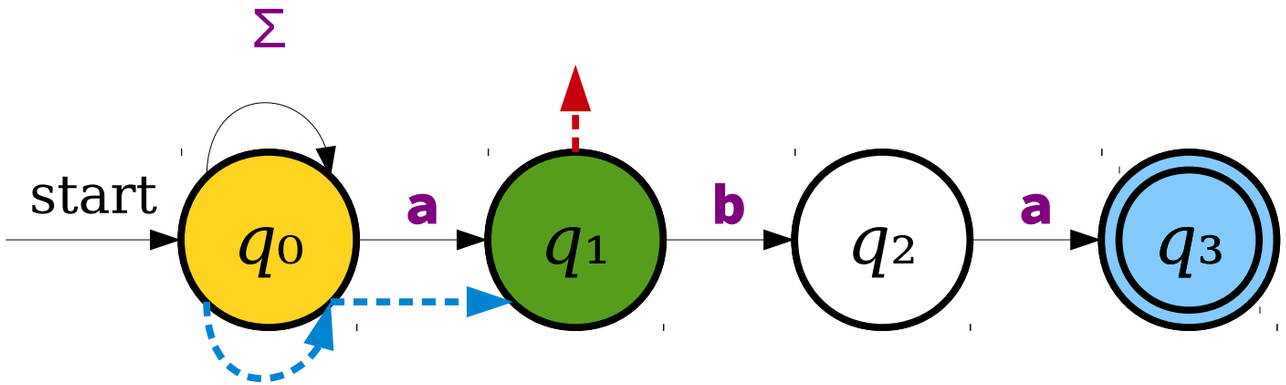

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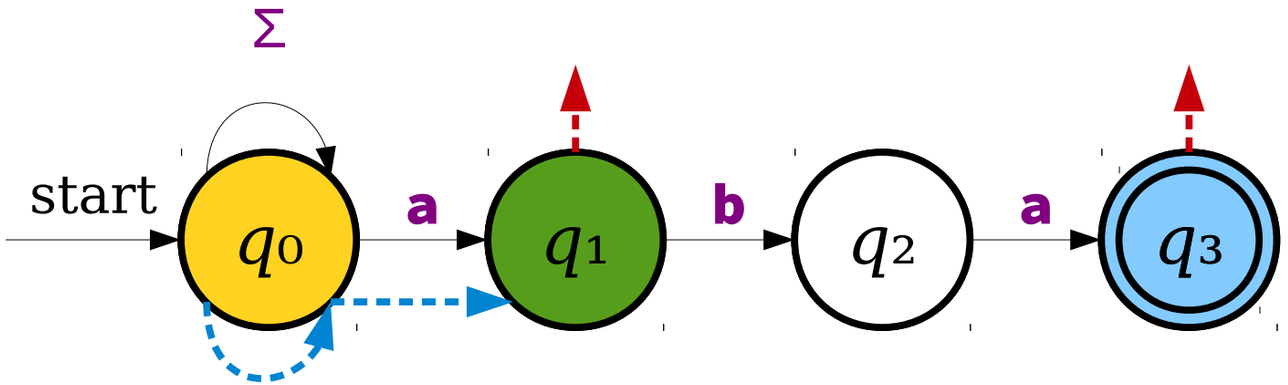

	a	b
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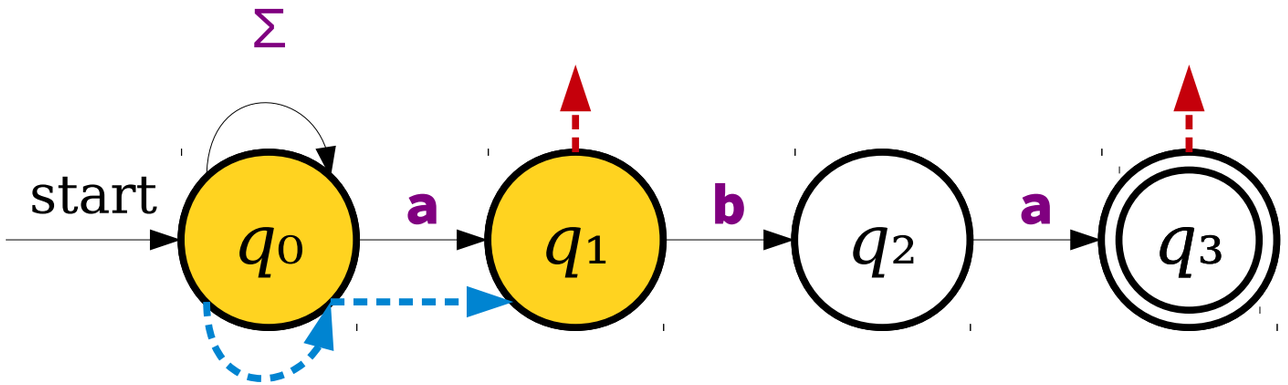

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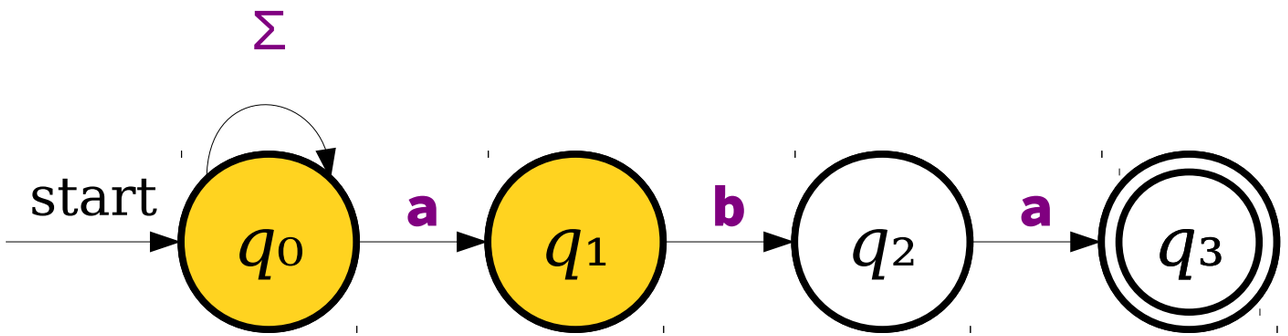

	a	b
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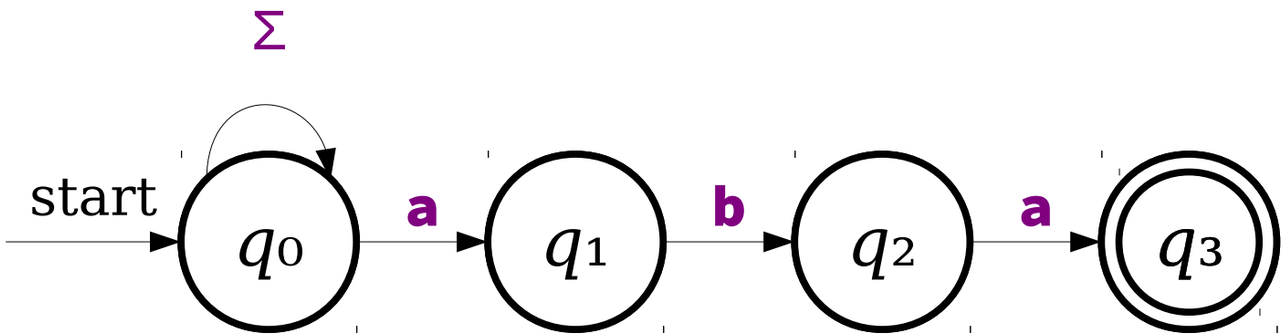

	a	b
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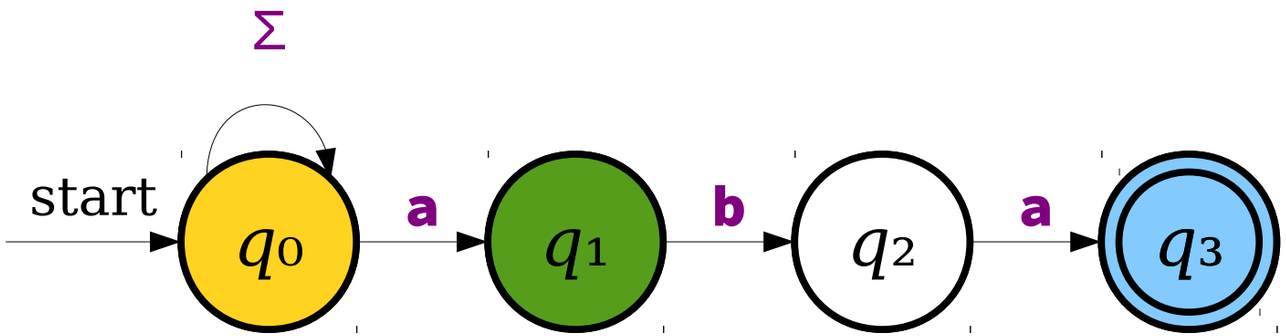

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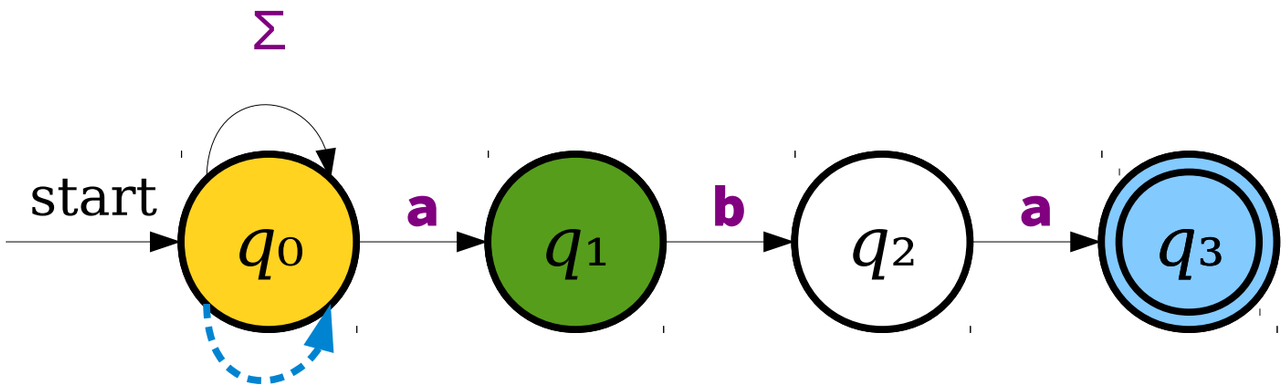

	a	b
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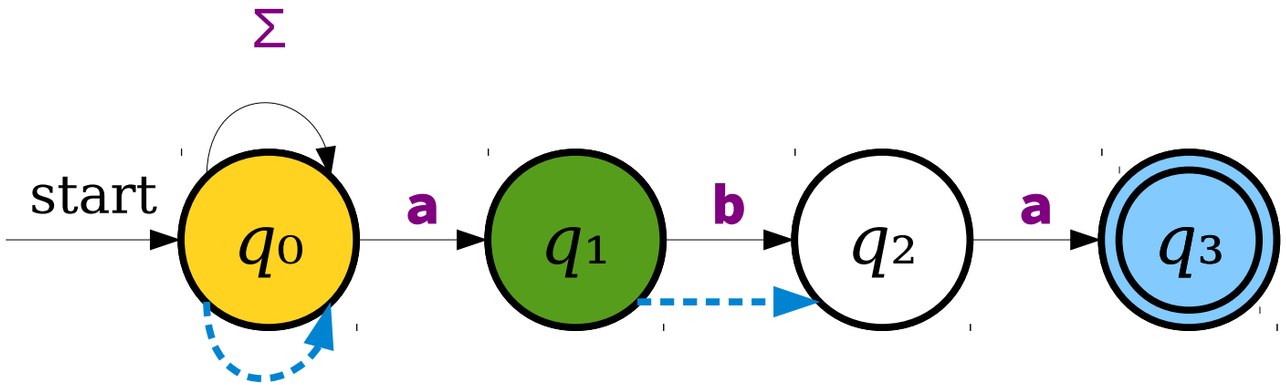

	a	b
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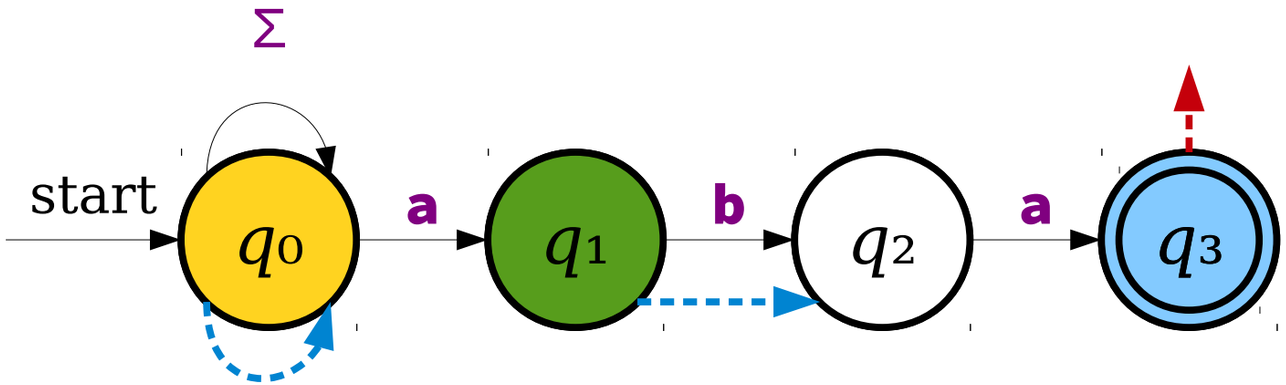

	a	b
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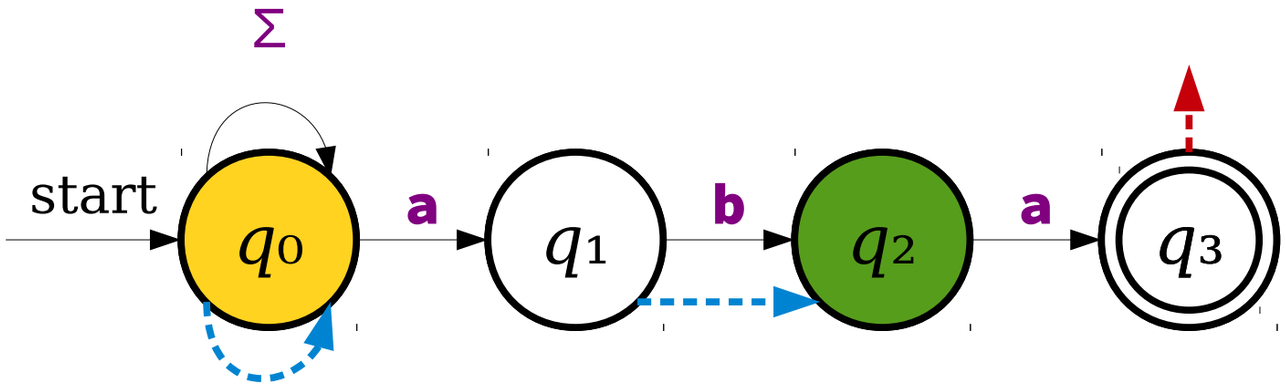

	a	b
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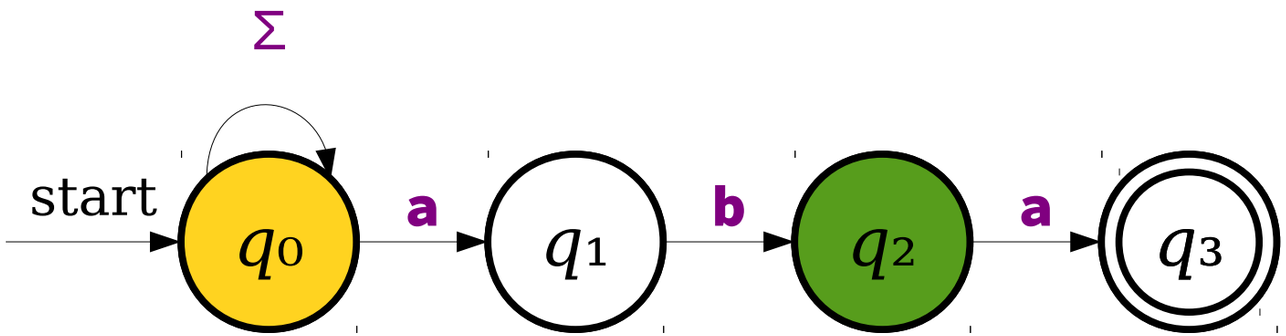

	a	b
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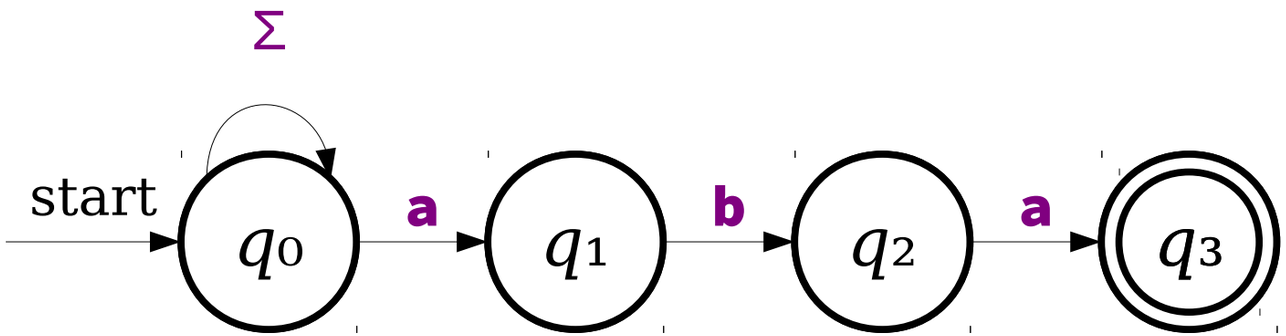

	a	b
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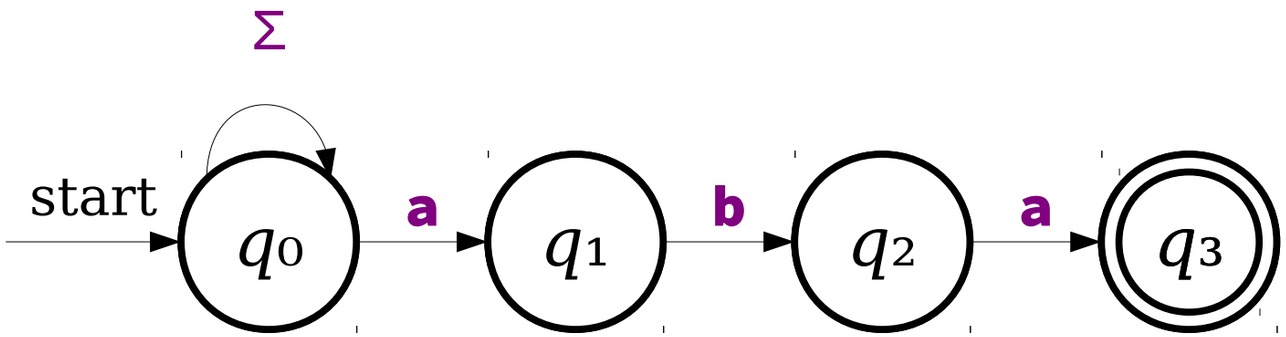

	a	b
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$\{q_0, q_1, q_3\}$	$\{q_0, q_1\}$	$\{q_0, q_2\}$

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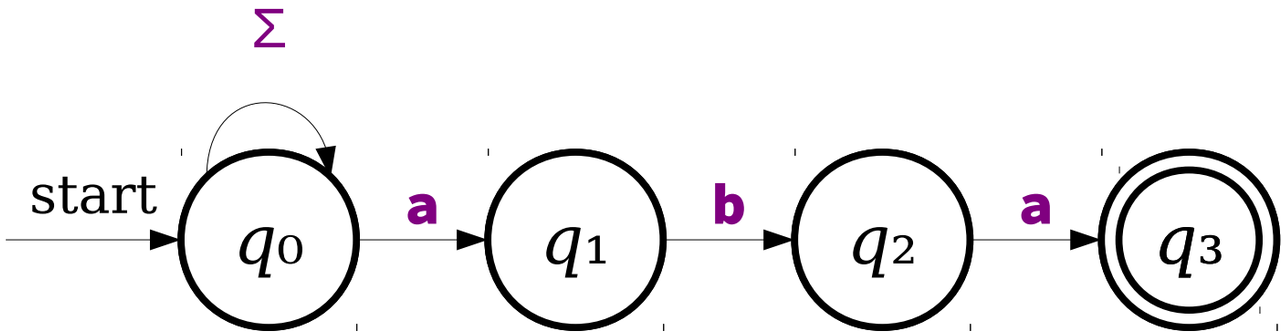
Think hard:
Remember that our NFA transition function says all the possibilities for states we can be in is $\wp(S)$. So if the original NFA had $n = |S|$ states, how many rows might this table need to have?



	a	b
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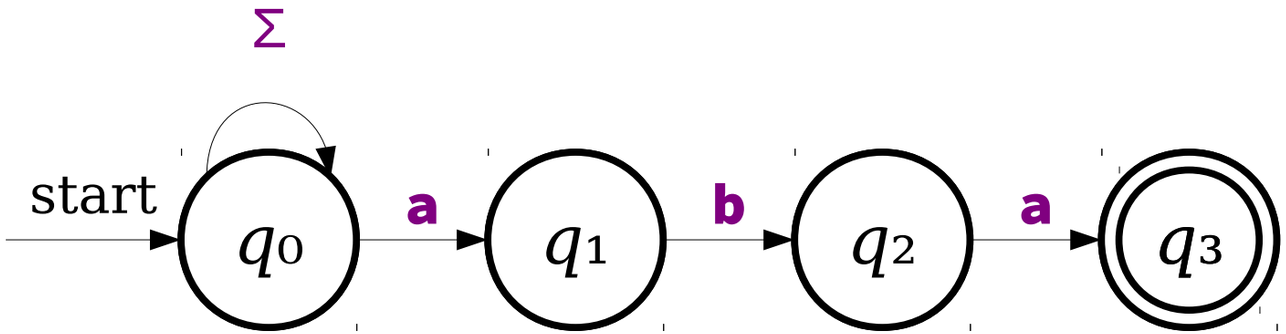
$\{q_0, q_1, q_3\}$	$\{q_0, q_1\}$	$\{q_0, q_2\}$

Think hard follow-up: It could need up to 2^n rows! For simplicity in this lecture example, we'll stop here, but you could imagine to fully formally define the transition function δ, we would need some token entries for all 2^n elements of $\wp(S)$.



	a	b
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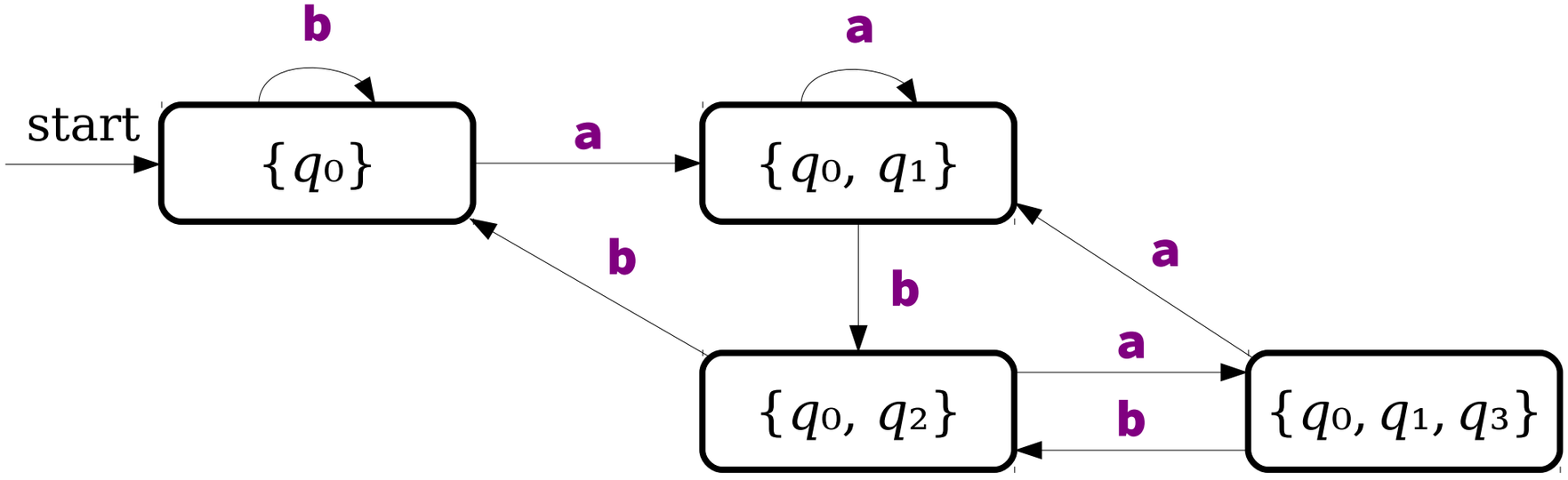

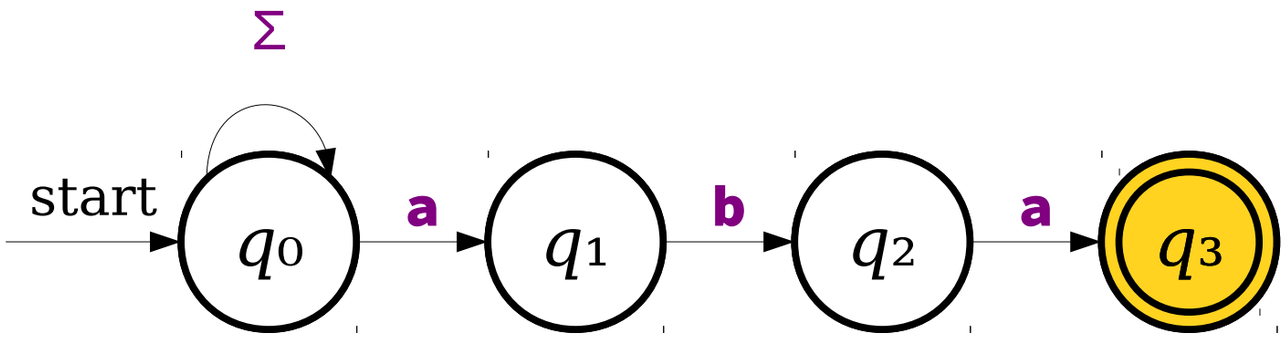

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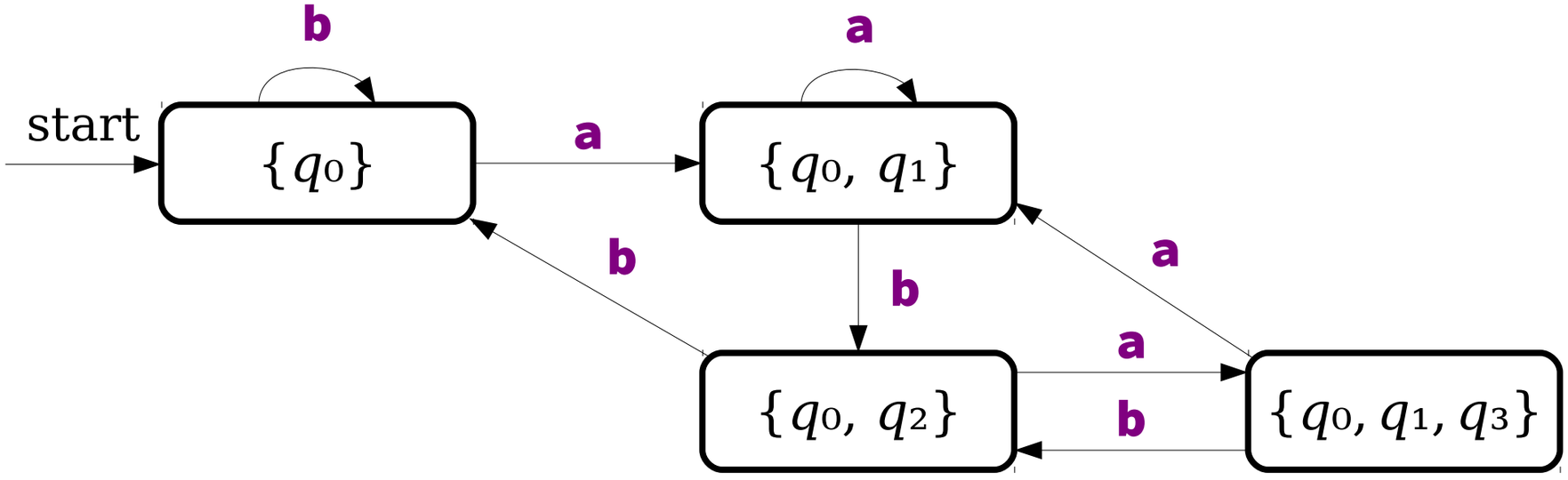

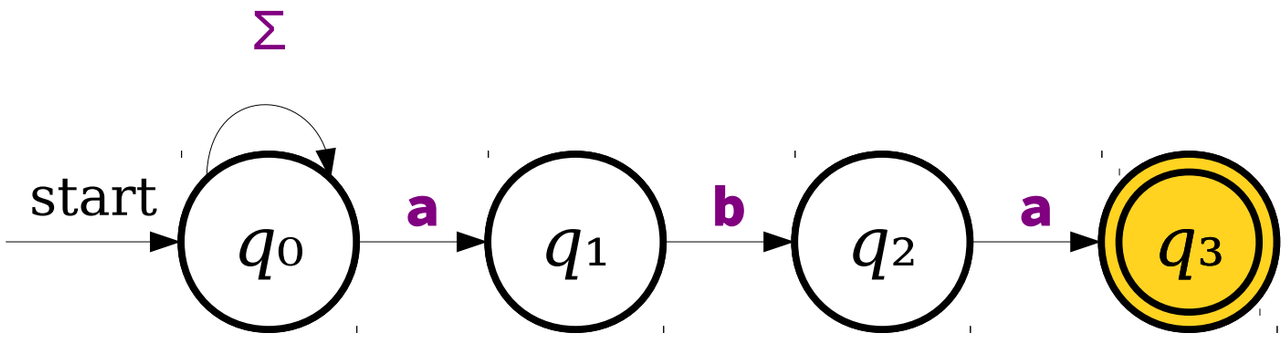

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$\{q_0, q_2\}$	$\{q_0, q_1, q_3\}$	$\{q_0\}$
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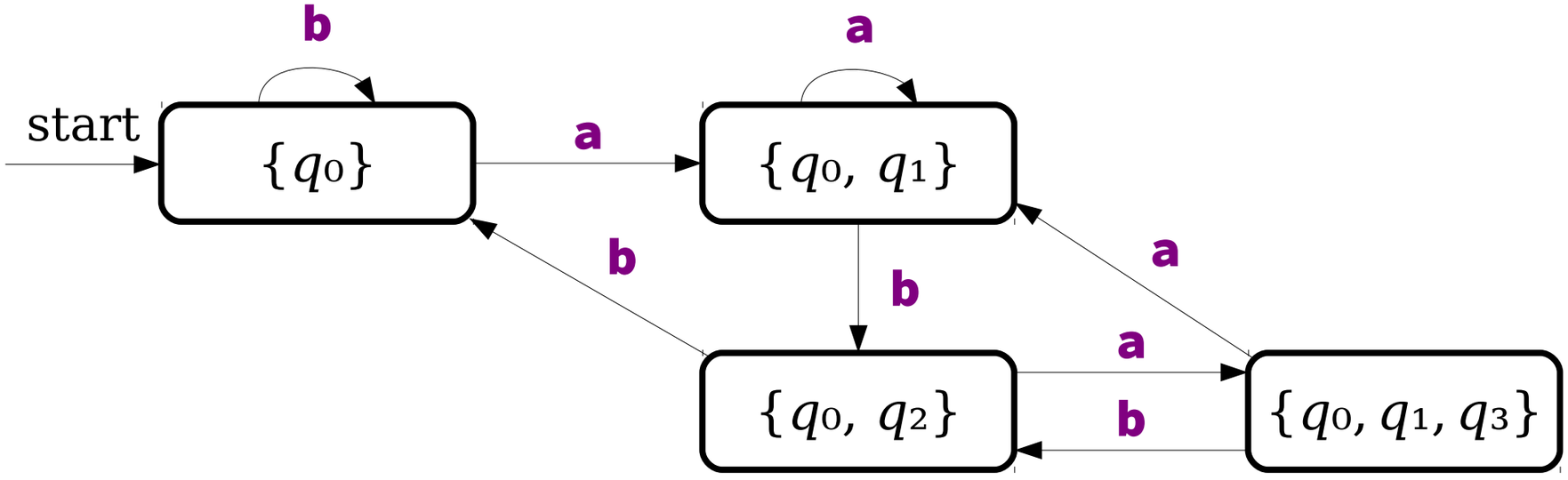

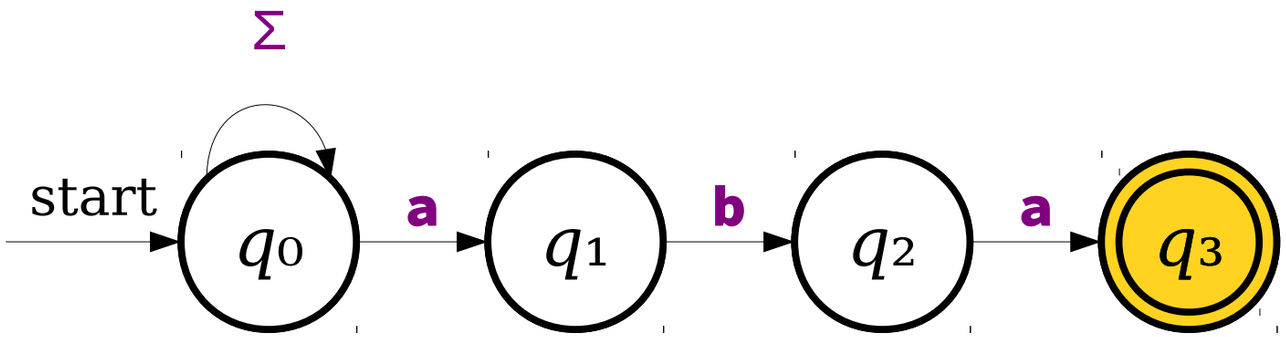

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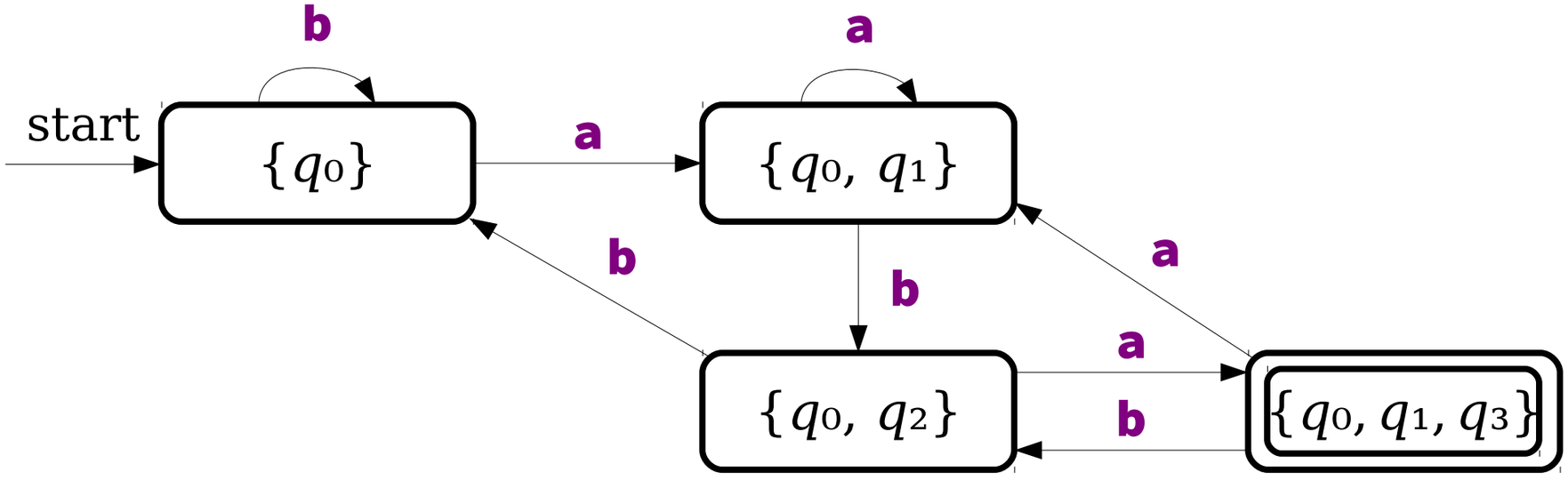

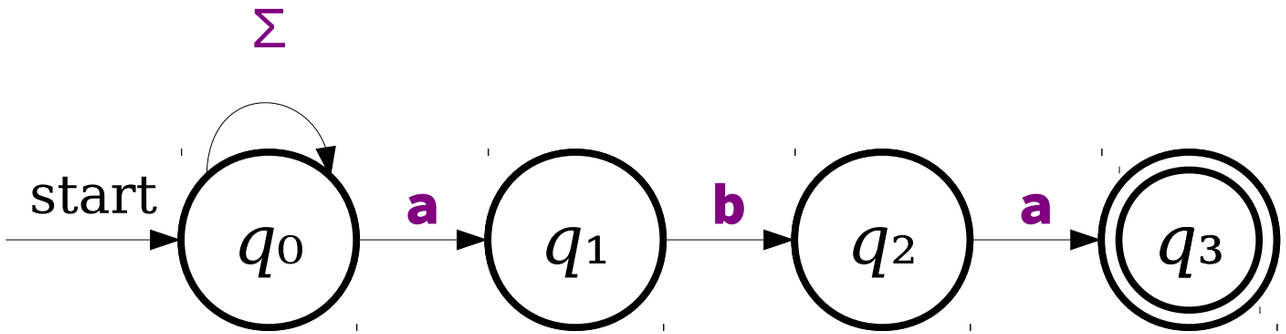

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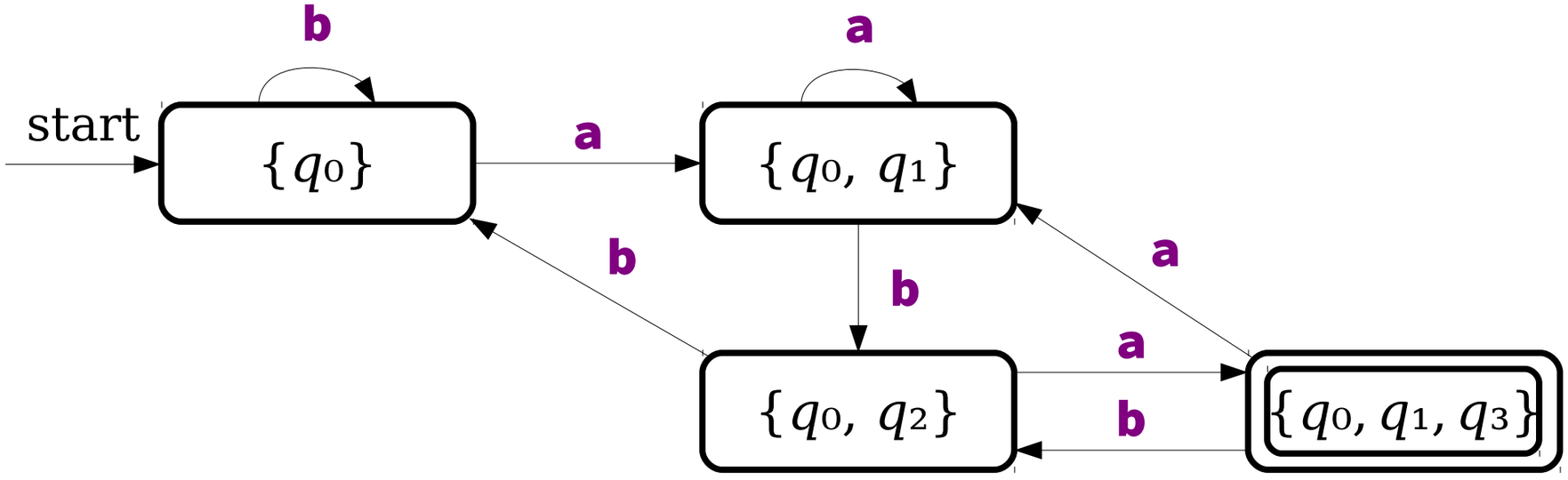

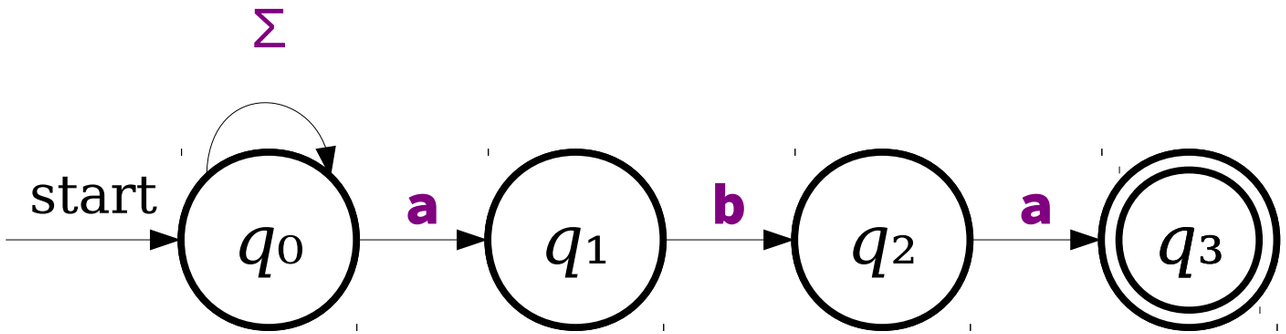

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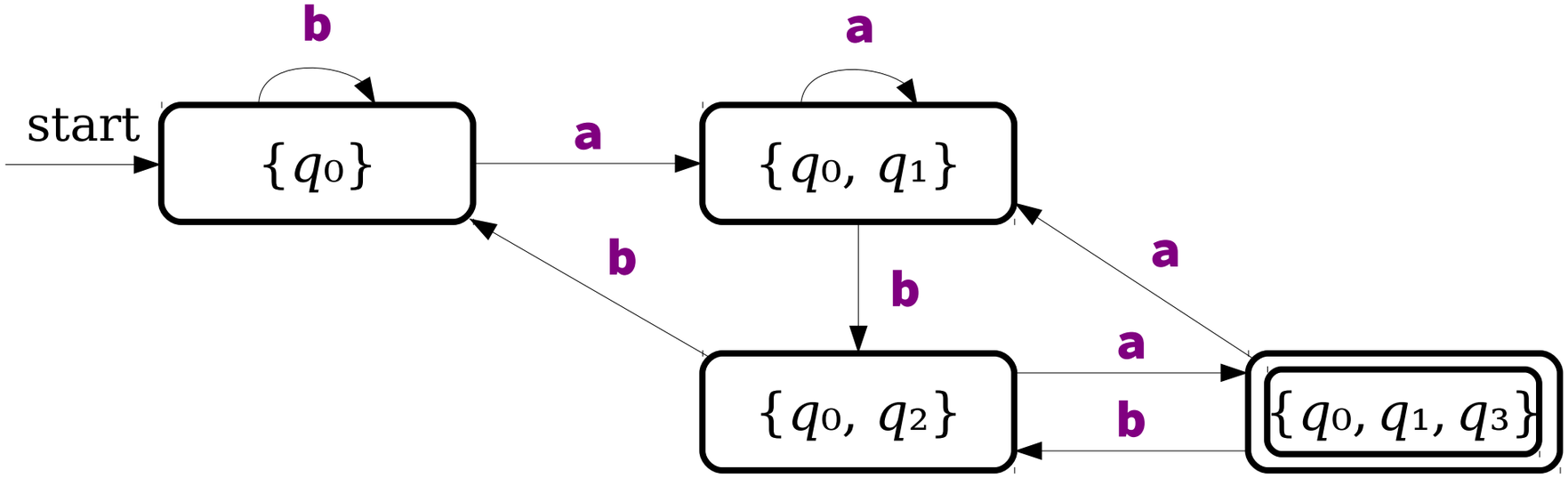

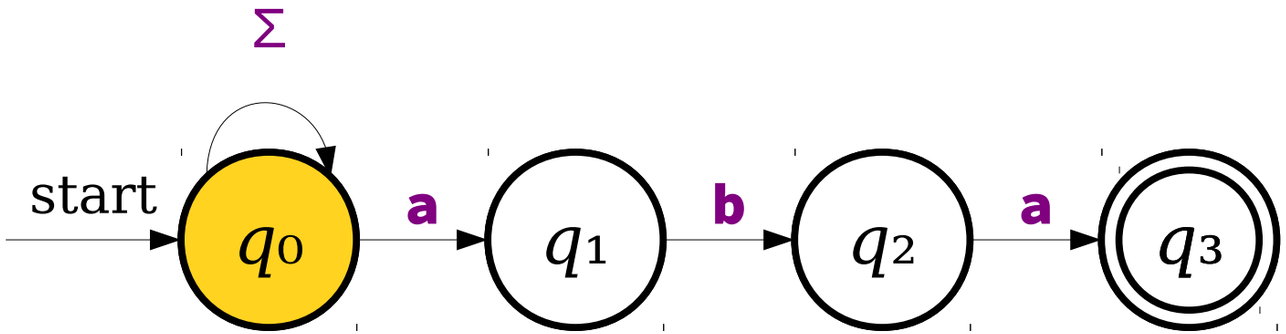

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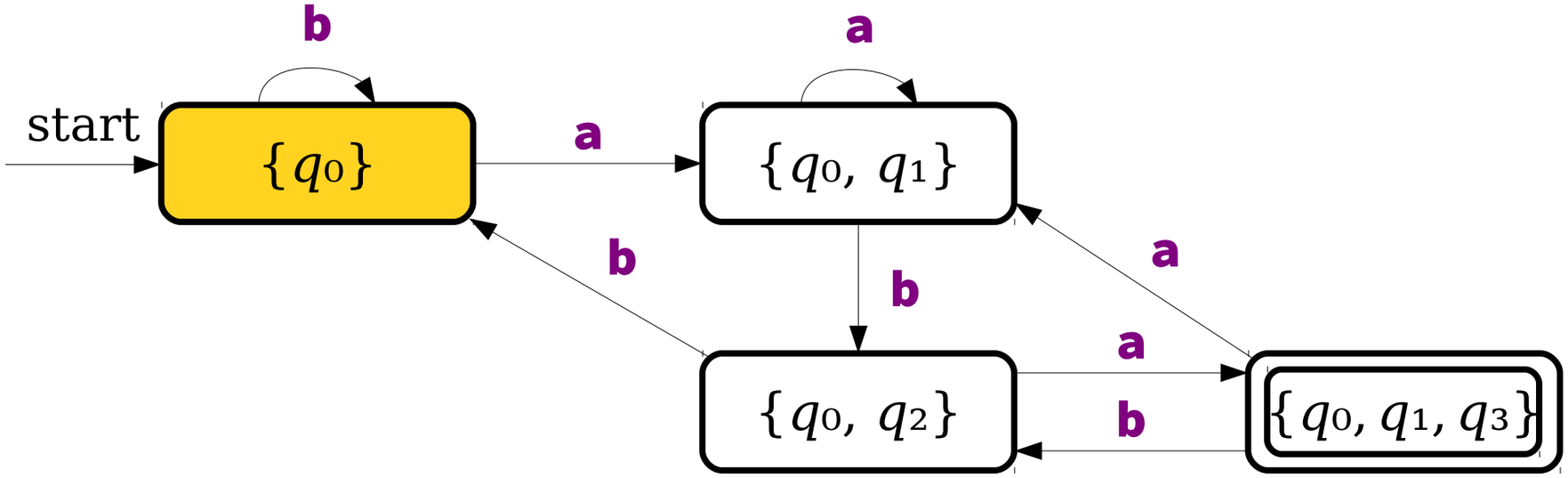

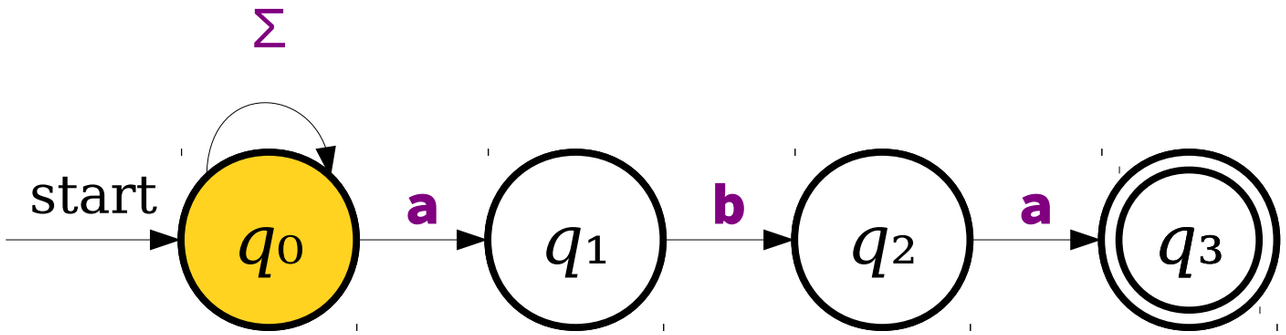
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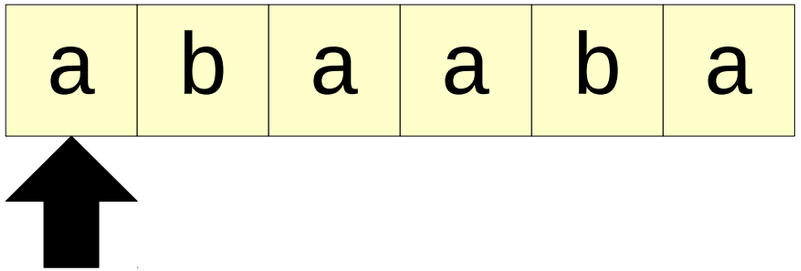
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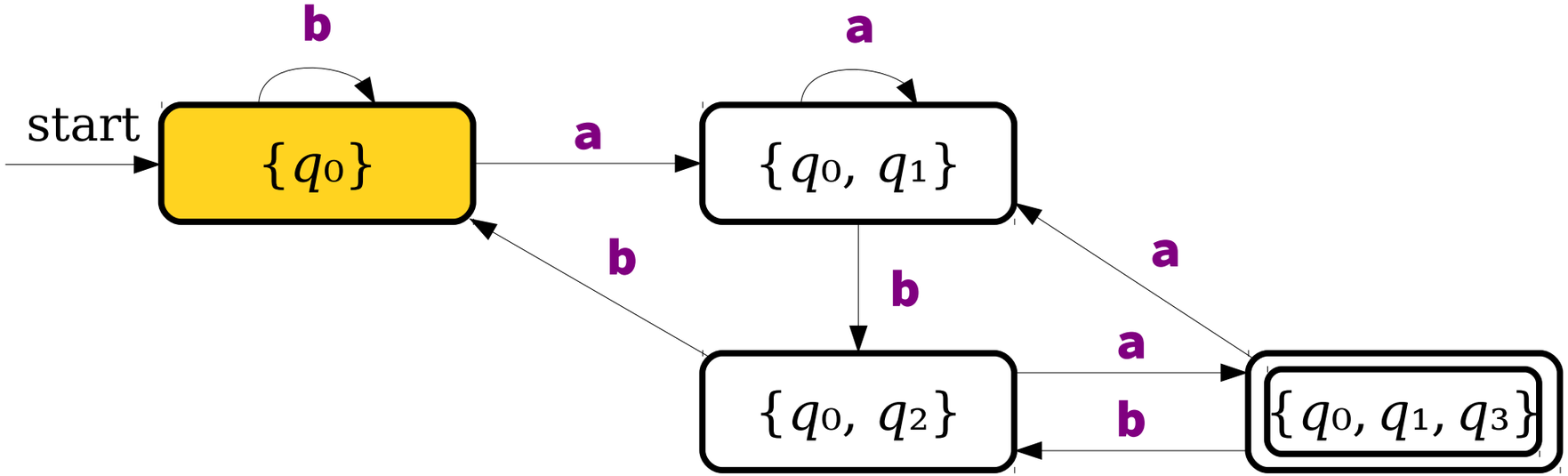
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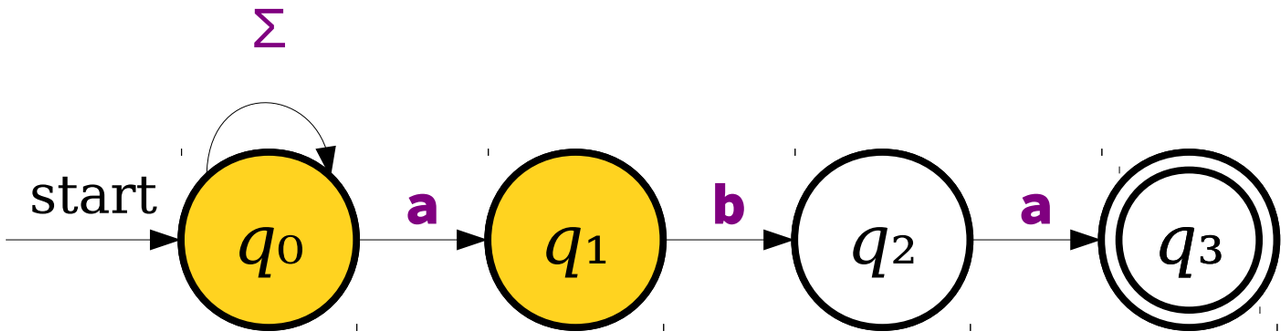
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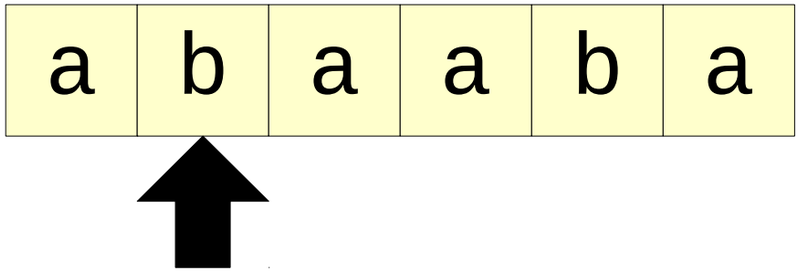
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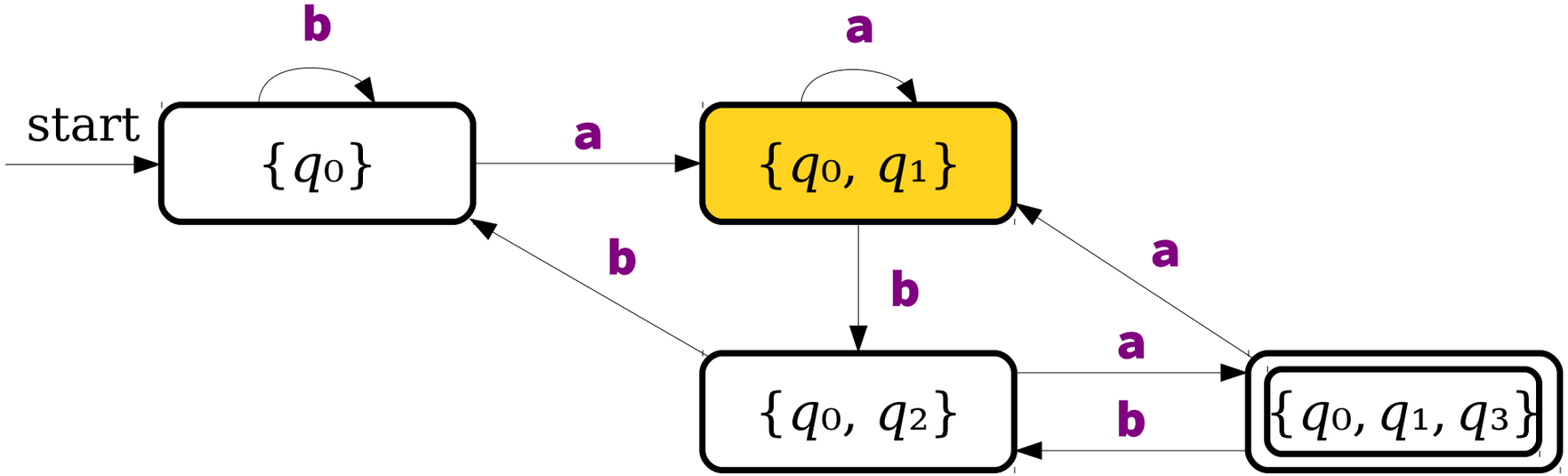
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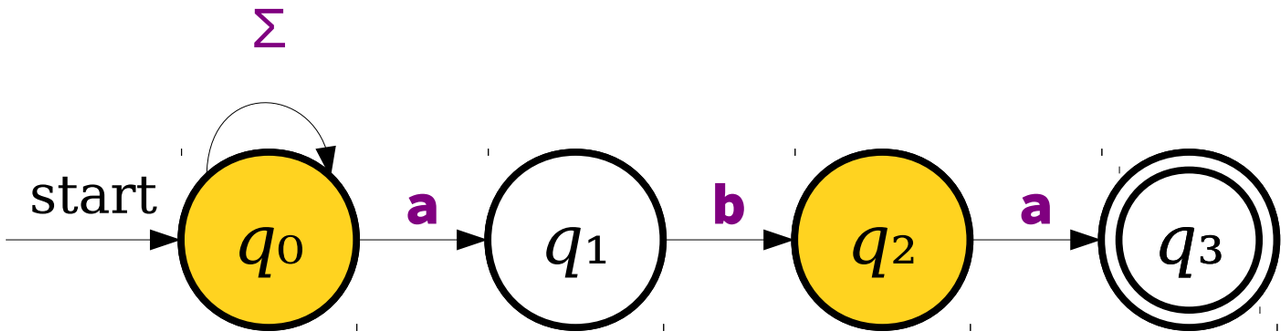
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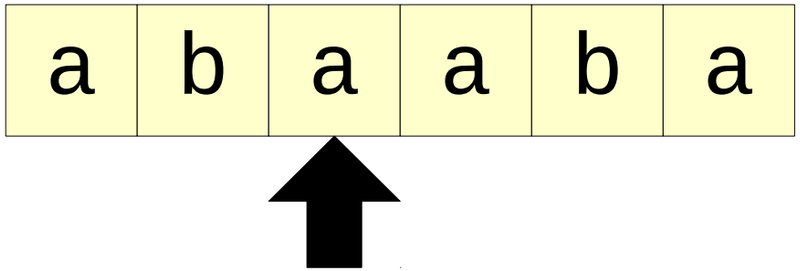
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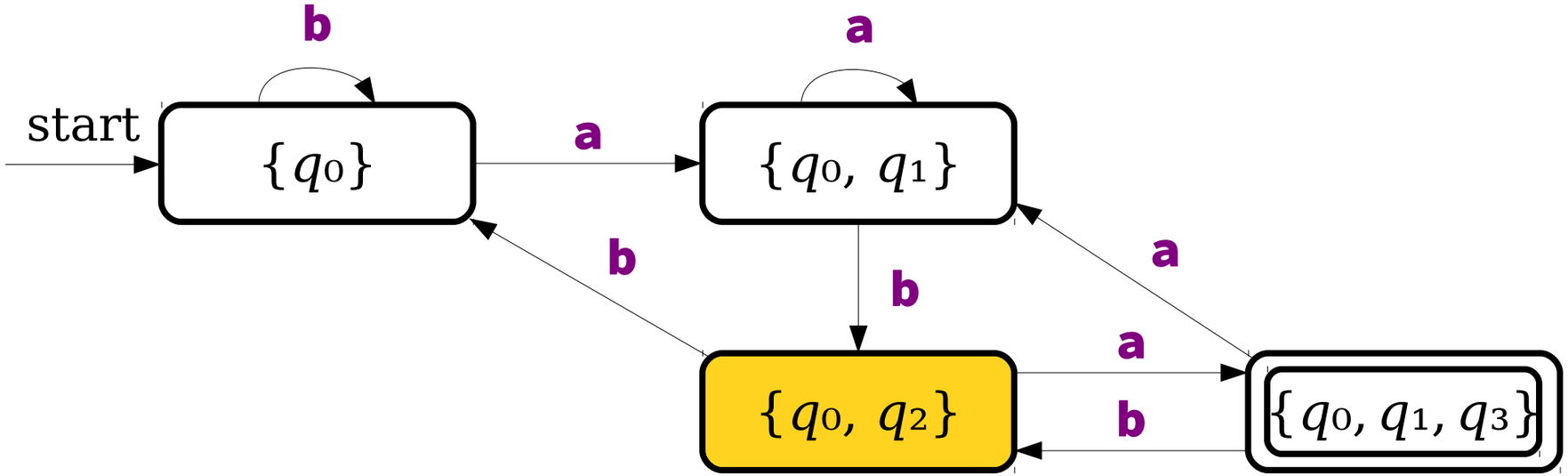
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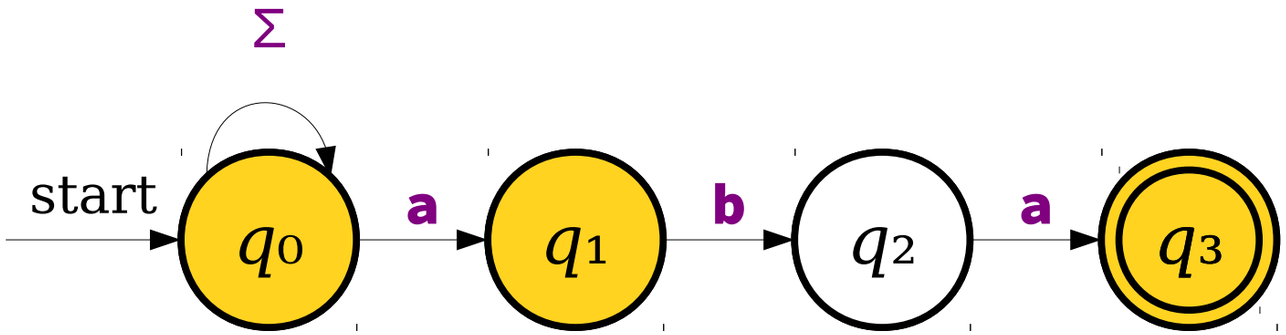
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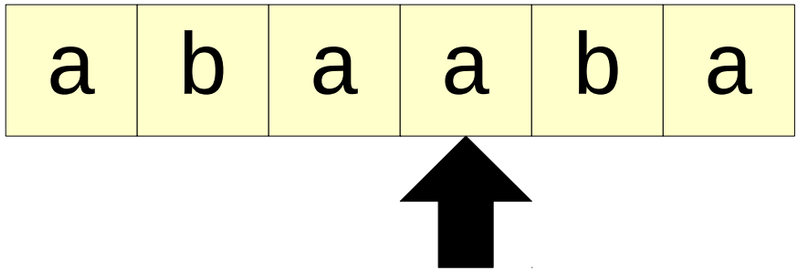
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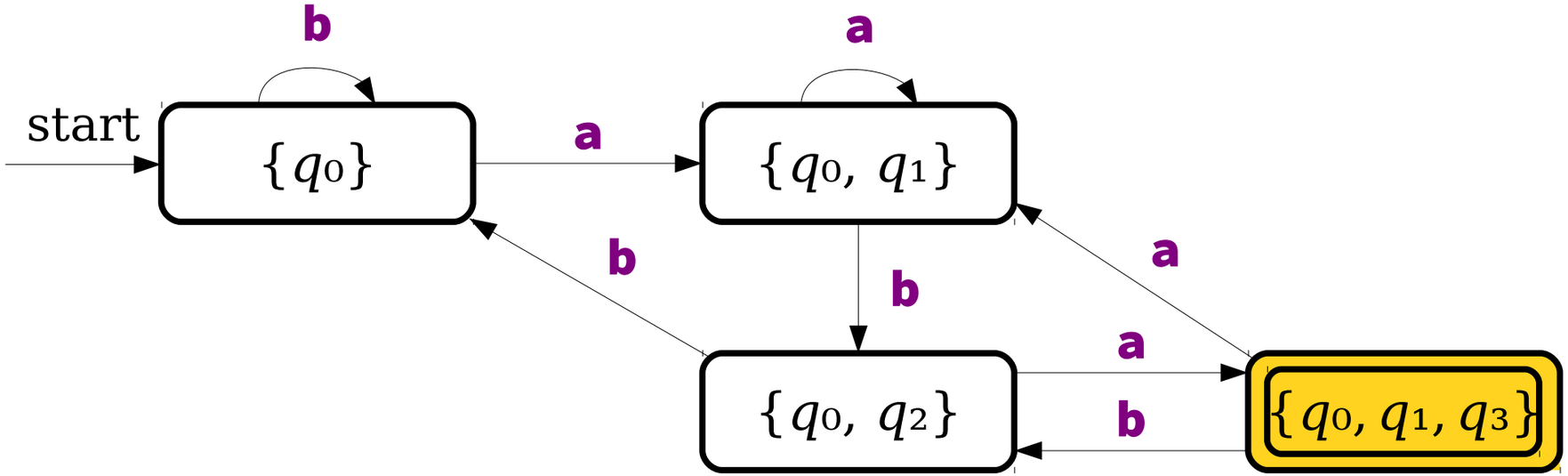
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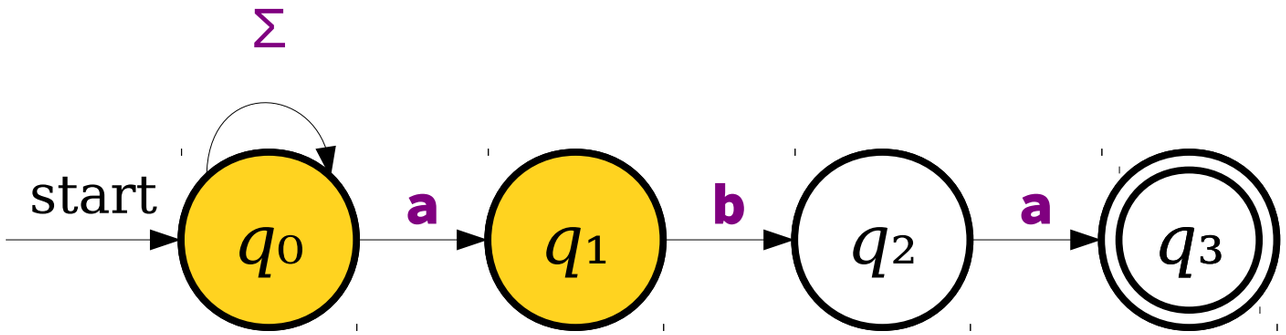
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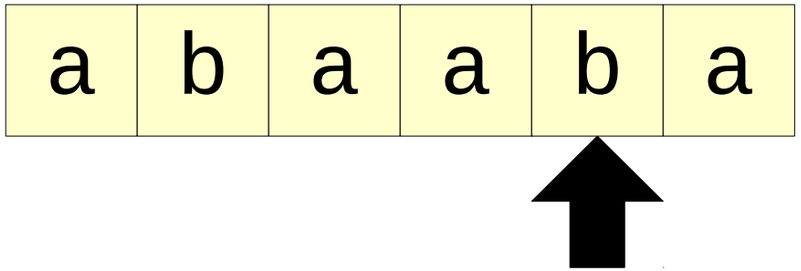
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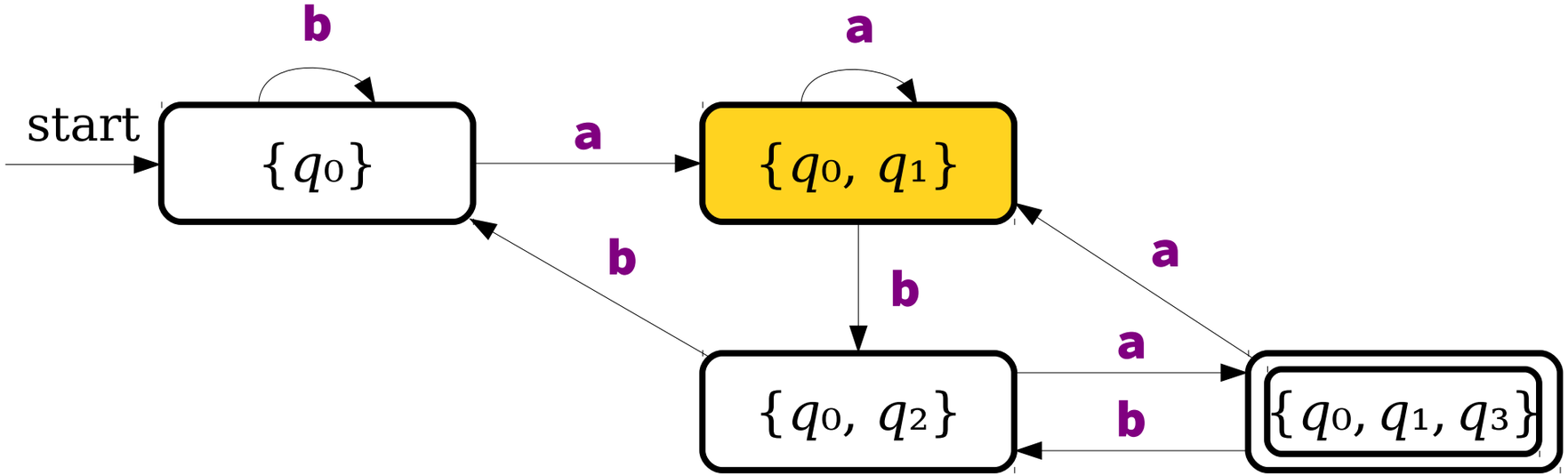
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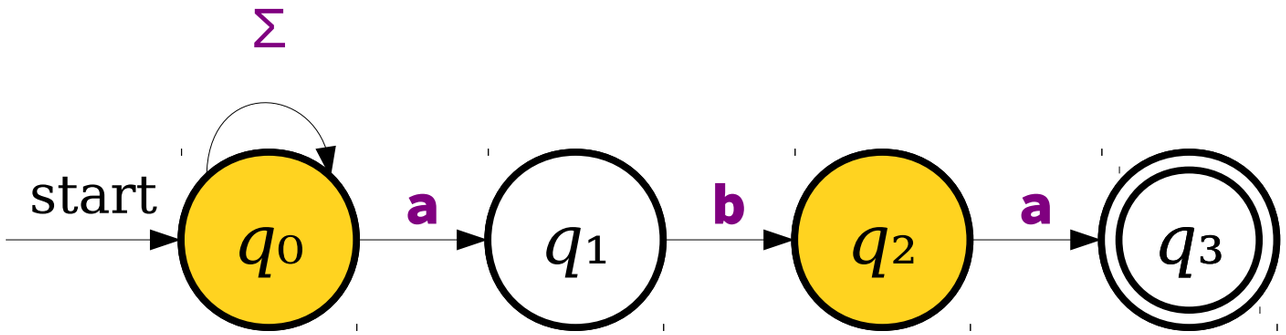
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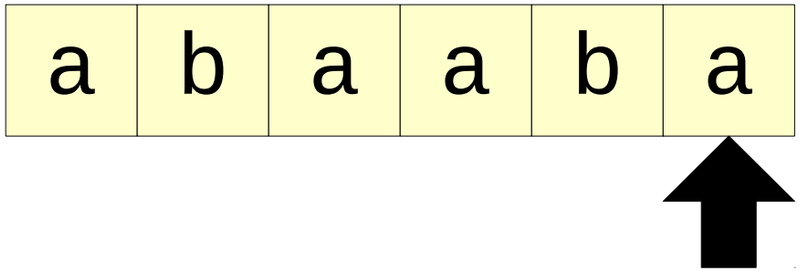
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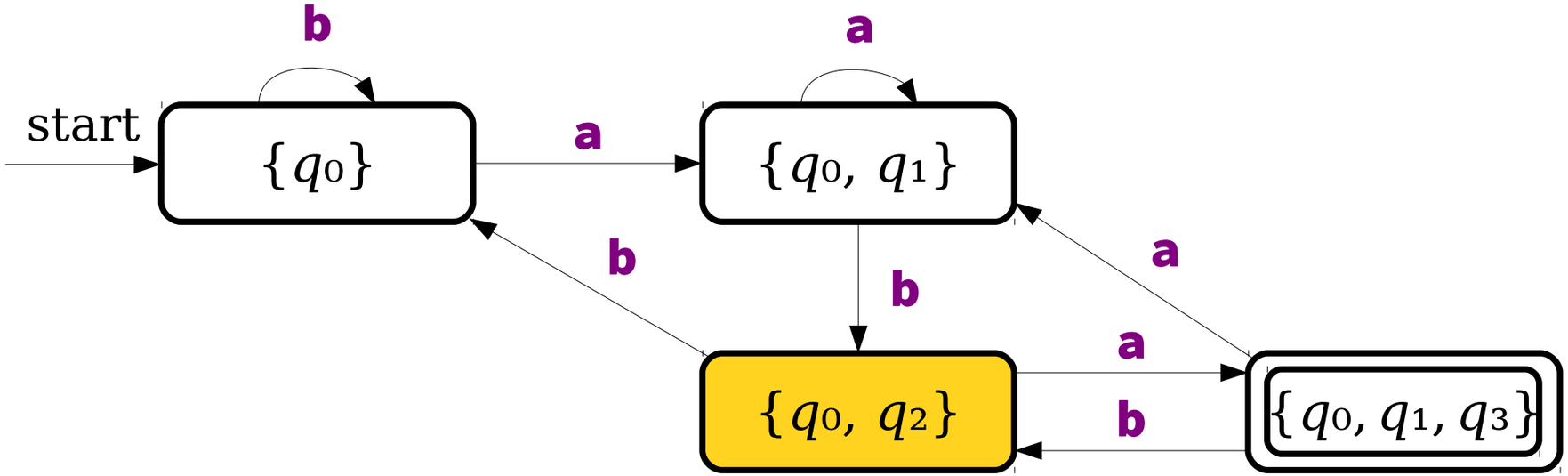
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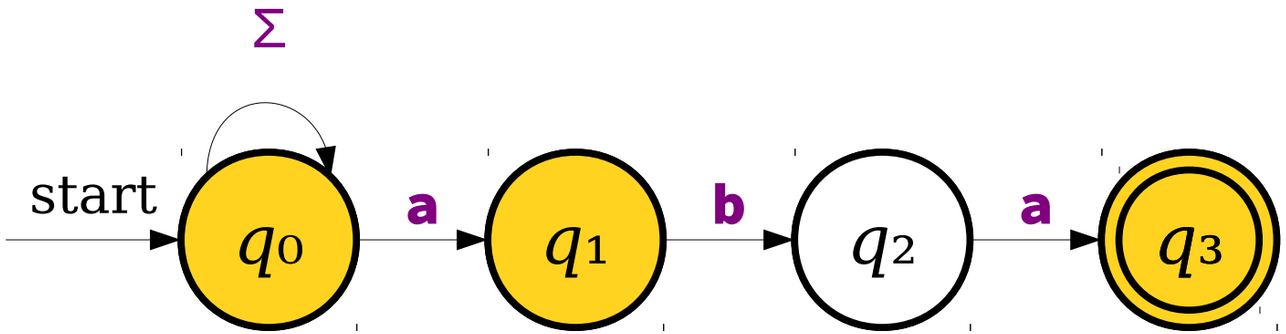
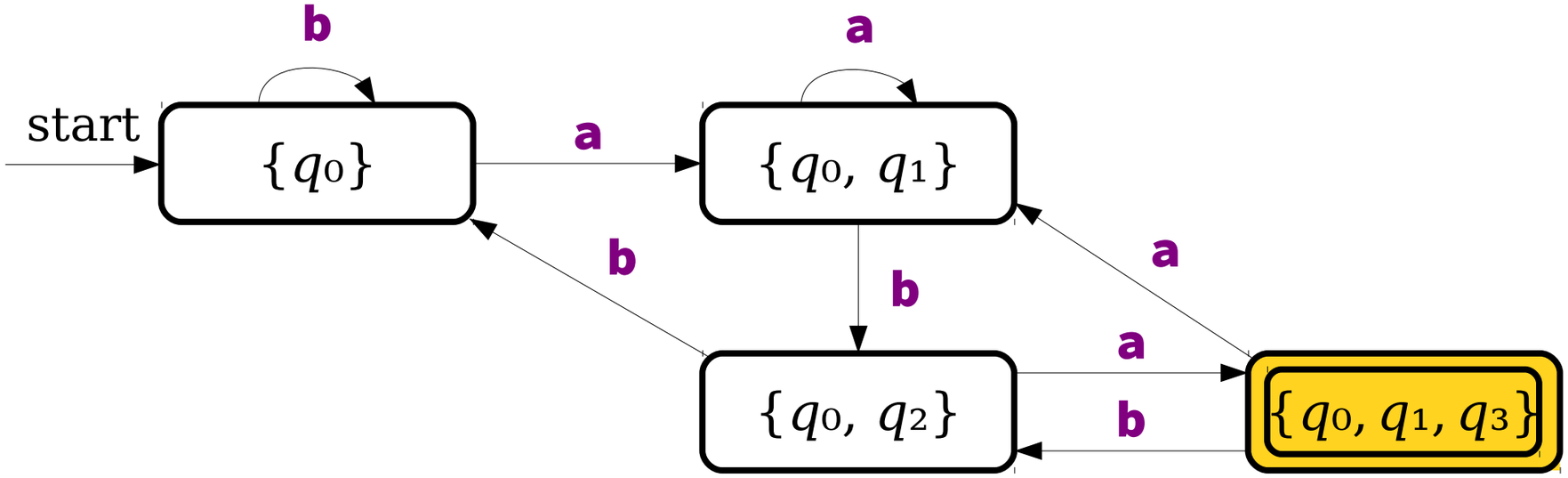


a	b	a	a	b	a



The Subset Construction

- This procedure for turning an NFA for a language L into a DFA for a language L is called the ***subset construction***.
  - It's sometimes called the ***powerset construction***; it's different names for the same thing!
- Intuitively:
  - Each state in the DFA corresponds to a set of states from the NFA.
  - Each transition in the DFA corresponds to what transitions would be taken in the NFA when using the massive parallel intuition.
  - The accepting states in the DFA correspond to which sets of states would be considered accepting in the NFA when using the massive parallel intuition.
- There's an online ***Guide to the Subset Construction*** with a more elaborate example involving ε-transitions and cases where the NFA dies; check that for more details.

The Subset Construction

- In converting an NFA to a DFA, the DFA's states correspond to sets of NFA states.
  - ***Useful fact:*** $|\wp(S)| = 2^{|S|}$ for any finite set S.
- So, in the worst-case, the construction can result in a DFA that is *exponentially larger* than the original NFA.
- ***Question to ponder:*** Can you find a family of languages that have NFAs of size n, but no DFAs of size less than 2^n?

Why This Matters

- We now have two perspectives on regular languages:
  - Regular languages are languages accepted by DFAs.
  - Regular languages are languages accepted by NFAs.
- We can now reason about the regular languages in two different ways.

Properties of Regular Languages

The Union of Two Languages

- If L_1 and L_2 are languages over the alphabet Σ, the language $L_1 \cup L_2$ is the language of all strings in at least one of the two languages.
- If L_1 and L_2 are regular languages, is $L_1 \cup L_2$?

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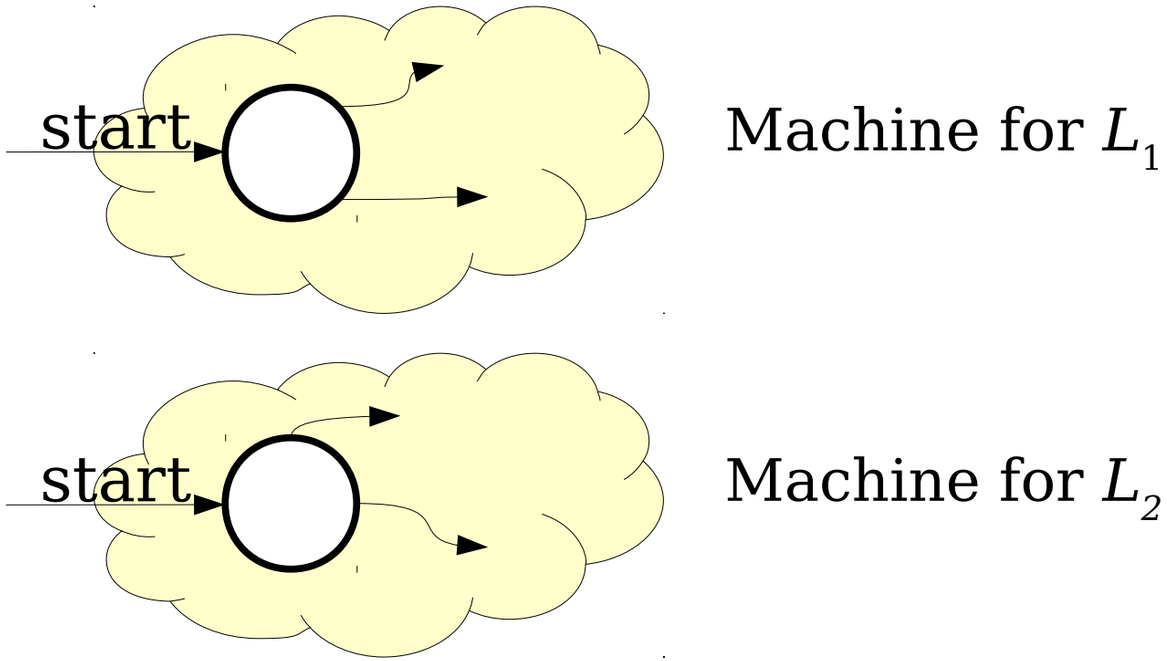

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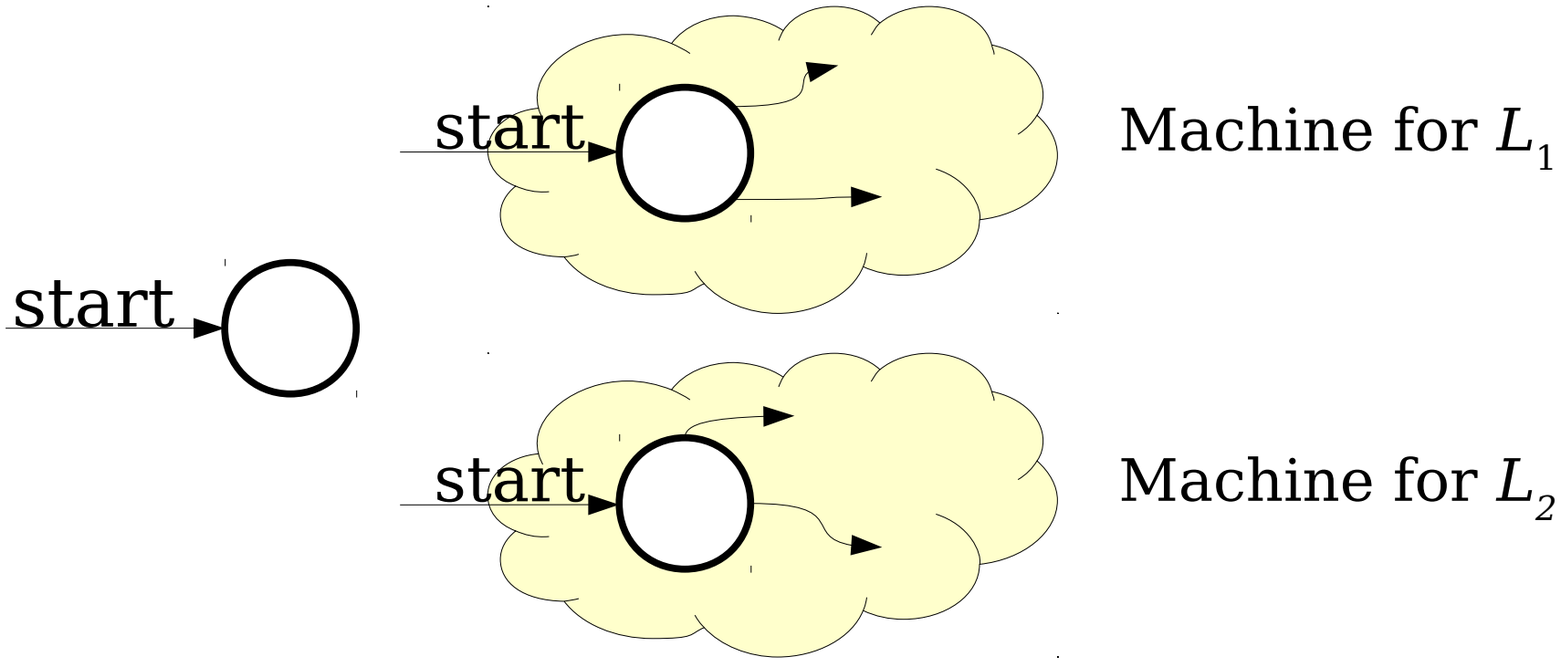

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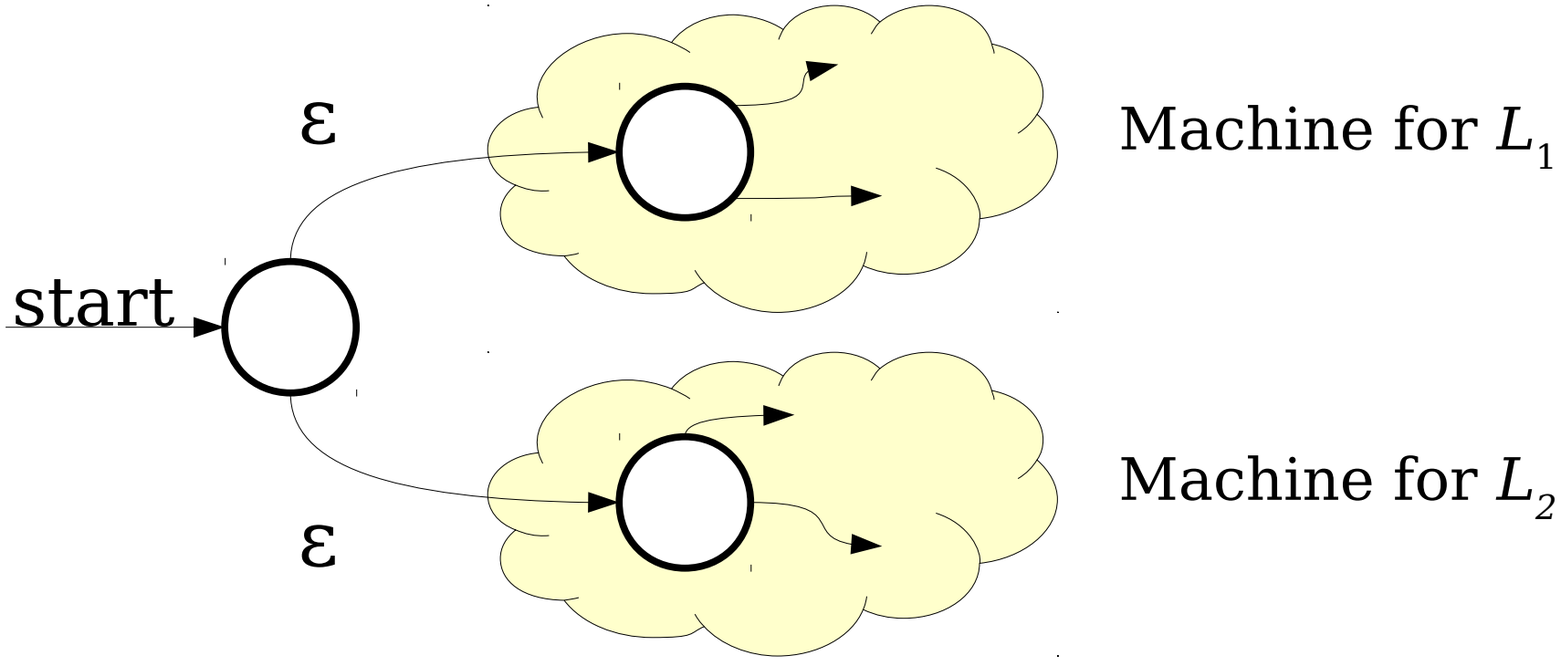

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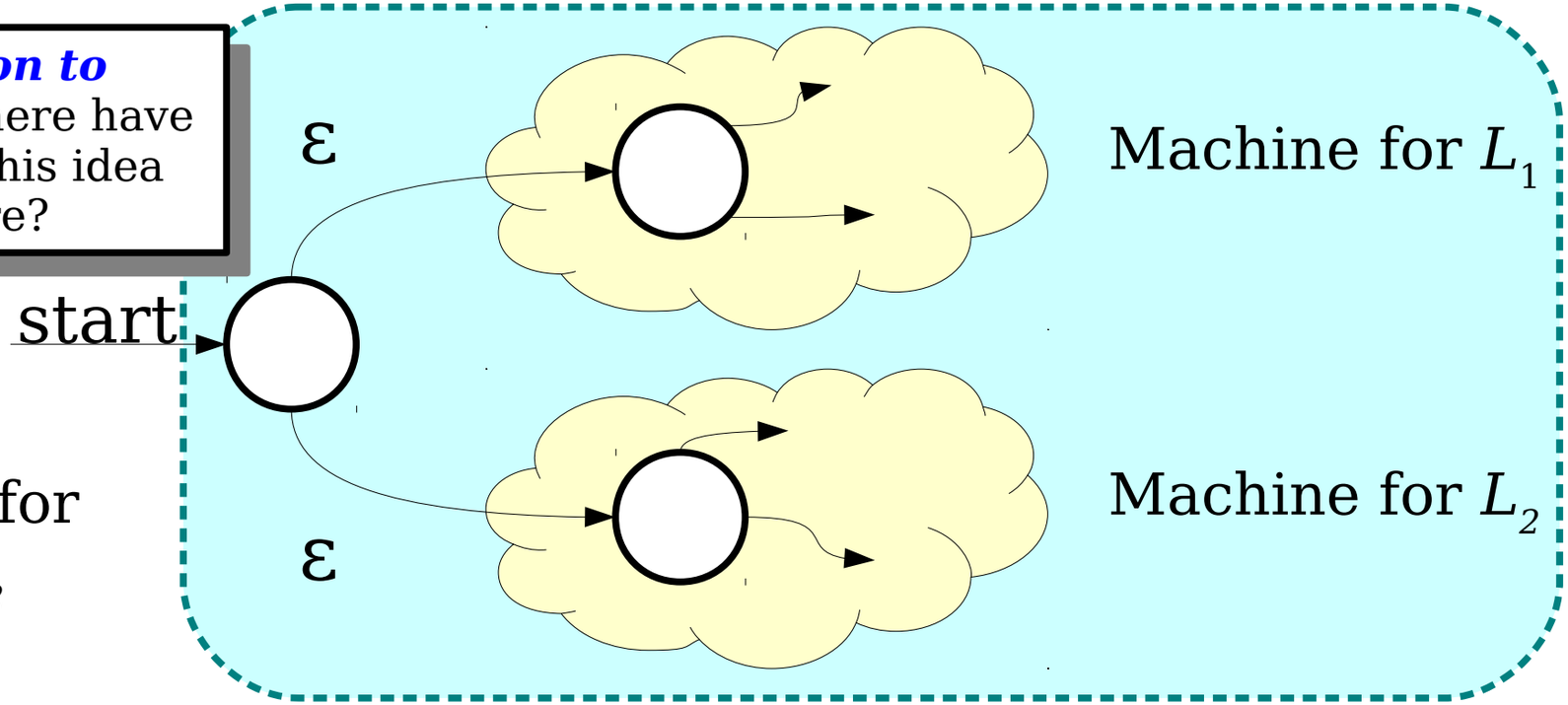
Question to ponder: where have you seen this idea before?



Machine for $L_1 \cup L_2$

The Intersection of Two Languages

- If L_1 and L_2 are languages over Σ, then $L_1 \cap L_2$ is the language of strings in both L_1 and L_2.
- Question: If L_1 and L_2 are regular, is $L_1 \cap L_2$ regular as well?

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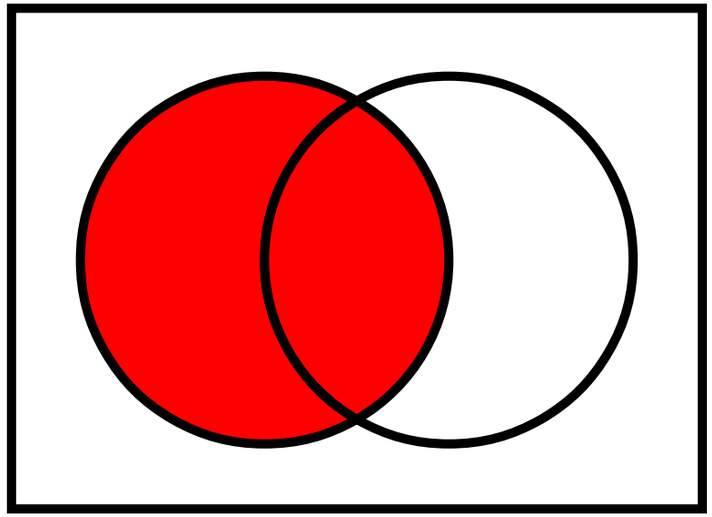
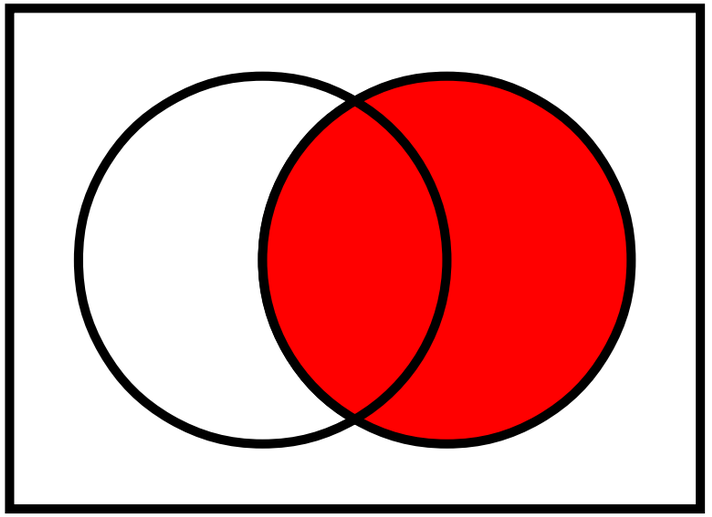


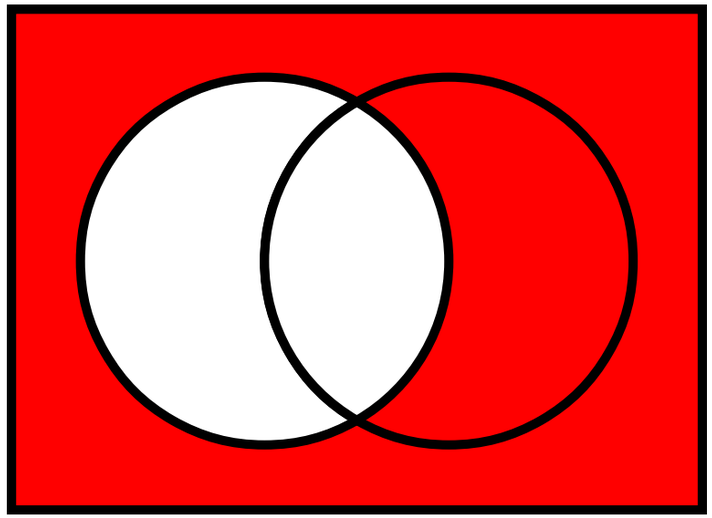
L_1



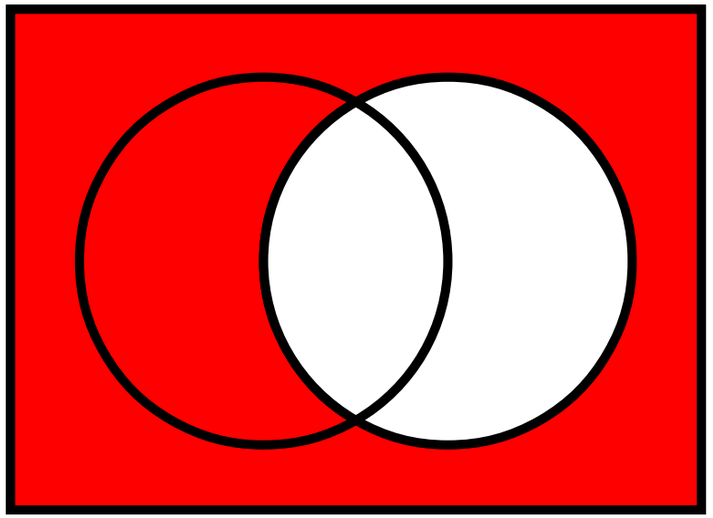
L_2

The Intersection of Two Languages

- If L_1 and L_2 are languages over Σ, then $L_1 \cap L_2$ is the language of strings in both L_1 and L_2.
- Question: If L_1 and L_2 are regular, is $L_1 \cap L_2$ regular as well?



$\overline{L}_1$



$\overline{L}_2$

The Intersection of Two Languages

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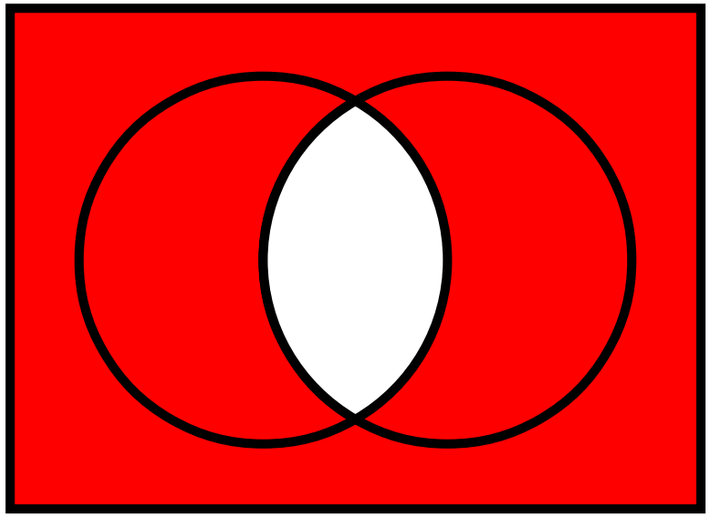
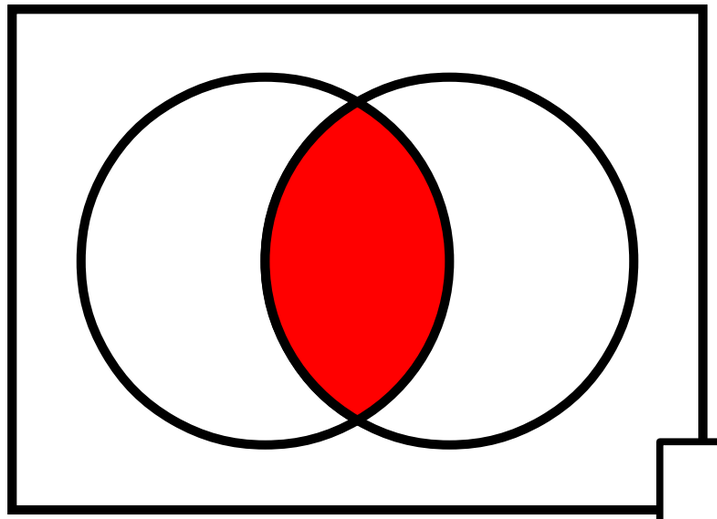


$$\overline{L}_1 \cup \overline{L}_2$$

The Intersection of Two Languages

- If L_1 and L_2 are languages over Σ, then $L_1 \cap L_2$ is the language of strings in both L_1 and L_2.
- Question: If L_1 and L_2 are regular, is $L_1 \cap L_2$ regular as well?



$$\overline{\overline{L_1} \cup \overline{L_2}}$$

Hey, it's De Morgan's laws!

Concatenation

String Concatenation

- If $w \in \Sigma^*$ and $x \in \Sigma^*$, the ***concatenation*** of w and x, denoted ***wx***, is the string formed by tacking all the characters of x onto the end of w.
- Example: if w = **quo** and x = **kka**, the concatenation wx = **quokka**.
- This is analogous to the + operator for strings in many programming languages.
- Some facts about concatenation:
  - The empty string ε is the ***identity element*** for concatenation:

$$w\varepsilon = \varepsilon w = w$$

  - Concatenation is ***associative***:

$$wxy = w(xy) = (wx)y$$

Concatenation

- The ***concatenation*** of two languages L_1 and L_2 over the alphabet Σ is the language

$$L_1L_2 = \{ wx \in \Sigma^* \mid w \in L_1 \wedge x \in L_2 \}$$

Concatenation Example

- Let Σ = { **a**, **b**, …, **z**, **A**, **B**, …, **Z** } and consider these languages over Σ:
  - ***Noun*** = { **Puppy**, **Rainbow**, **Whale**, … }
  - ***Verb*** = { **Hugs**, **Juggles**, **Loves**, … }
  - ***The*** = { **The** }
- The language ***TheNounVerbTheNoun*** is
  - { **ThePuppyHugsTheWhale**, **TheWhaleLovesTheRainbow**, **TheRainbowJugglesTheRainbow**, … }

Concatenation

- The ***concatenation*** of two languages L_1 and L_2 over the alphabet Σ is the language

$$L_1L_2 = \{ wx \in \Sigma^* \mid w \in L_1 \wedge x \in L_2 \}$$

- Two views of L_1L_2:
  - The set of all strings that can be made by concatenating a string in L_1 with a string in L_2.
  - The set of strings that can be split into two pieces: a piece from L_1 and a piece from L_2.

This is closely related to, but different than, the Cartesian product.

Question to ponder: In what ways are concatenations similar to Cartesian products? In what ways are they different?

Concatenating Regular Languages

- If L_1 and L_2 are regular languages, is L_1L_2?
- Intuition – can we split a string w into two strings xy such that $x \in L_1$ and $y \in L_2$?
- ***Idea:***

Concatenating Regular Languages

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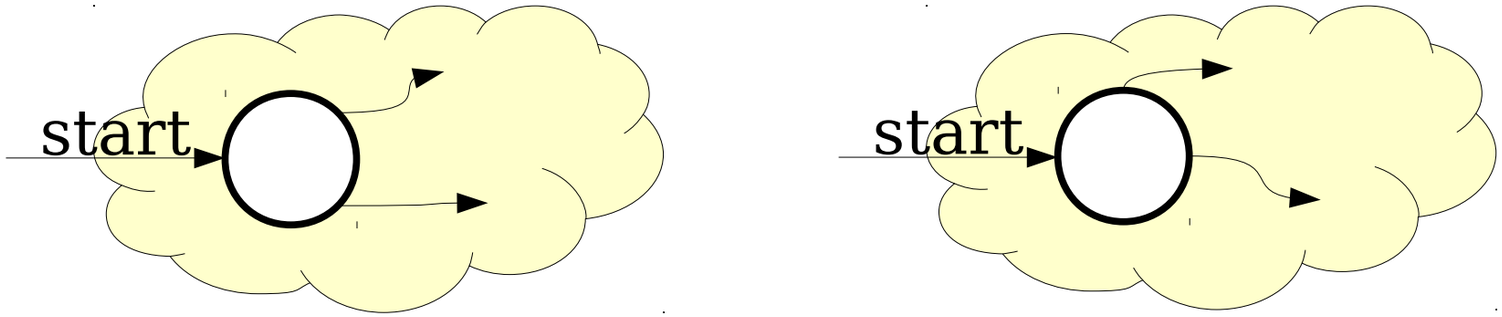

Machine for L_1 Machine for L_2

Concatenating Regular Languages

- If L_1 and L_2 are regular languages, is L_1L_2?
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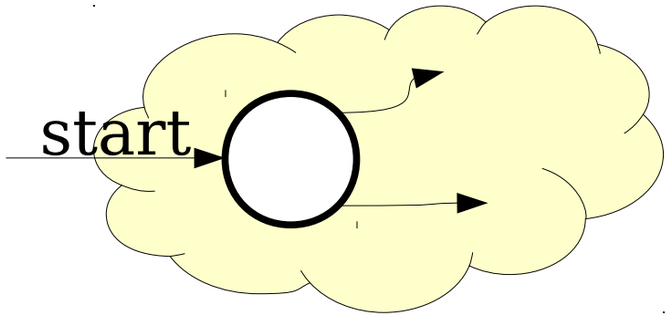
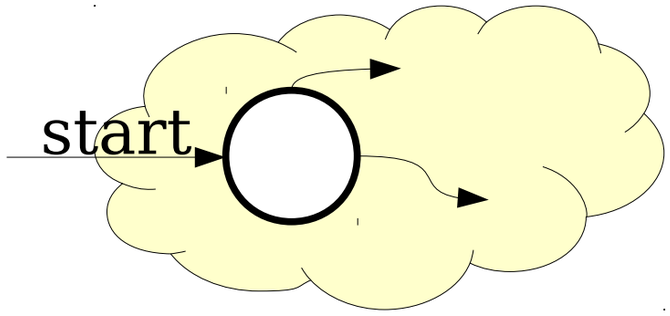


Machine for L_1

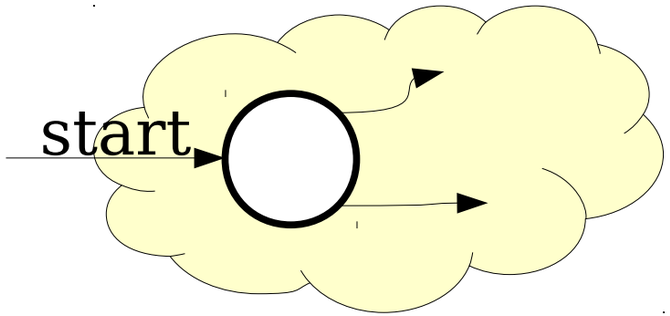


Machine for L_2

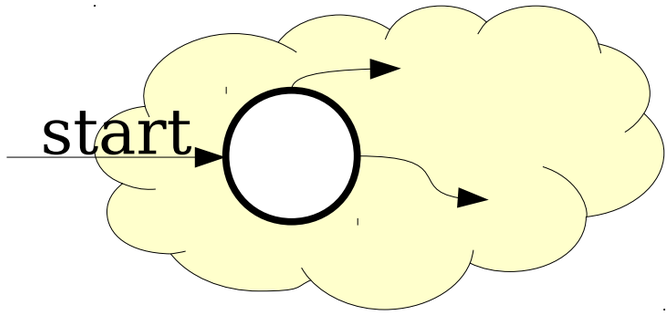
b	**o**	**o**	**k**	**k**	**e**	**e**	**p**	**e**	**r**

Concatenating Regular Languages

- If L_1 and L_2 are regular languages, is L_1L_2?
- Intuition – can we split a string w into two strings xy such that $x \in L_1$ and $y \in L_2$?
- ***Idea:***



Machine for L_1

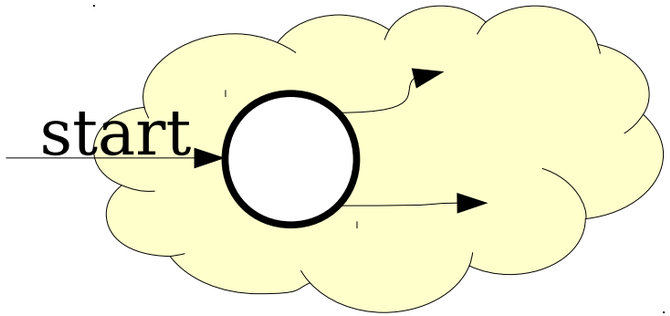


Machine for L_2

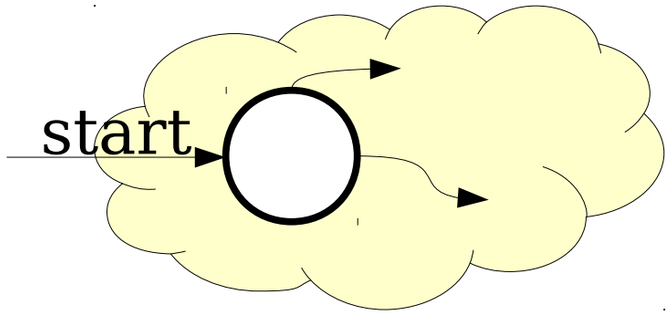
b	**o**	**o**	**k**	**k**	**e**	**e**	**p**	**e**	**r**

Concatenating Regular Languages

- If L_1 and L_2 are regular languages, is L_1L_2?
- Intuition – can we split a string w into two strings xy such that $x \in L_1$ and $y \in L_2$?
- ***Idea:***



Machine for L_1



Machine for L_2

b	o	o	k

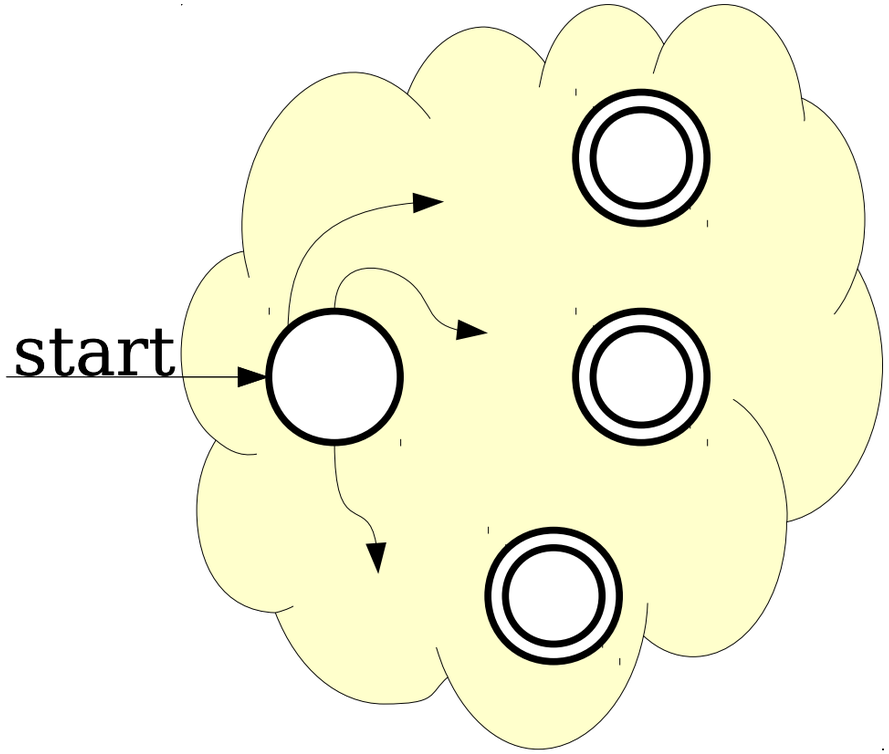
k	e	e	p	e	r

Concatenating Regular Languages

- If L_1 and L_2 are regular languages, is L_1L_2?
- Intuition – can we split a string w into two strings xy such that $x \in L_1$ and $y \in L_2$?
- ***Idea***:
  - Run a DFA/NFA for L_1 on w.
  - Whenever it reaches an accepting state, optionally hand the rest of w to a DFA/NFA for L_2.
  - If the automaton for L_2 accepts the rest, $w \in L_1L_2$.
  - If the automaton for L_2 rejects the remainder, the split was incorrect.

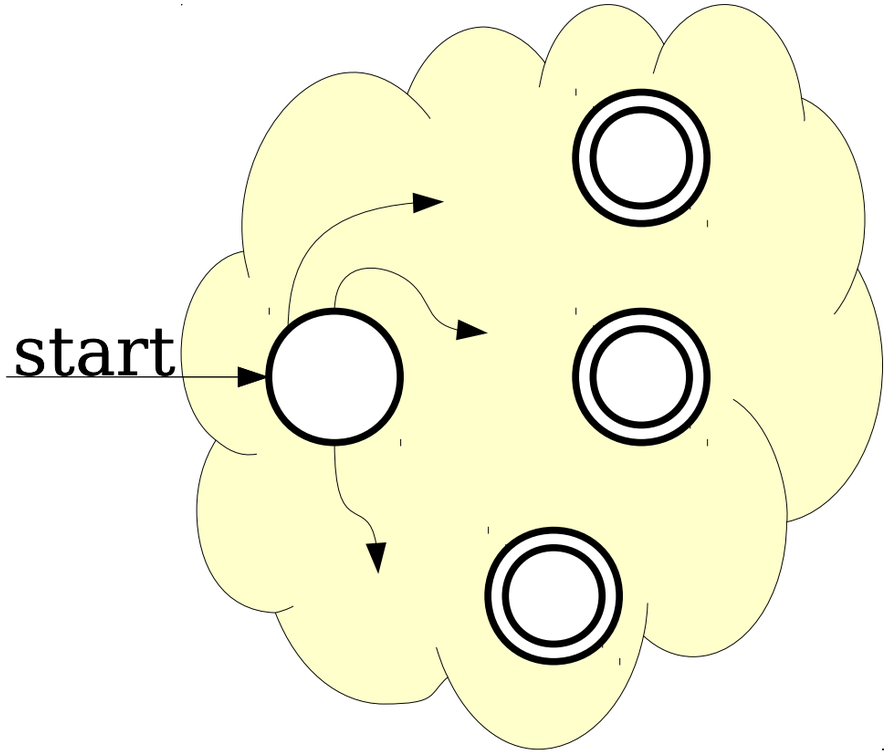
Concatenating Regular Languages

Concatenating Regular Languages

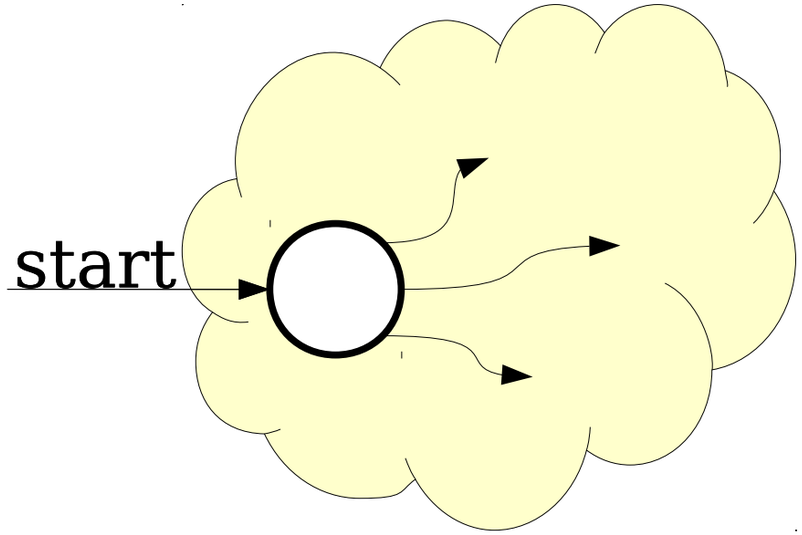


Machine for L_1

Concatenating Regular Languages

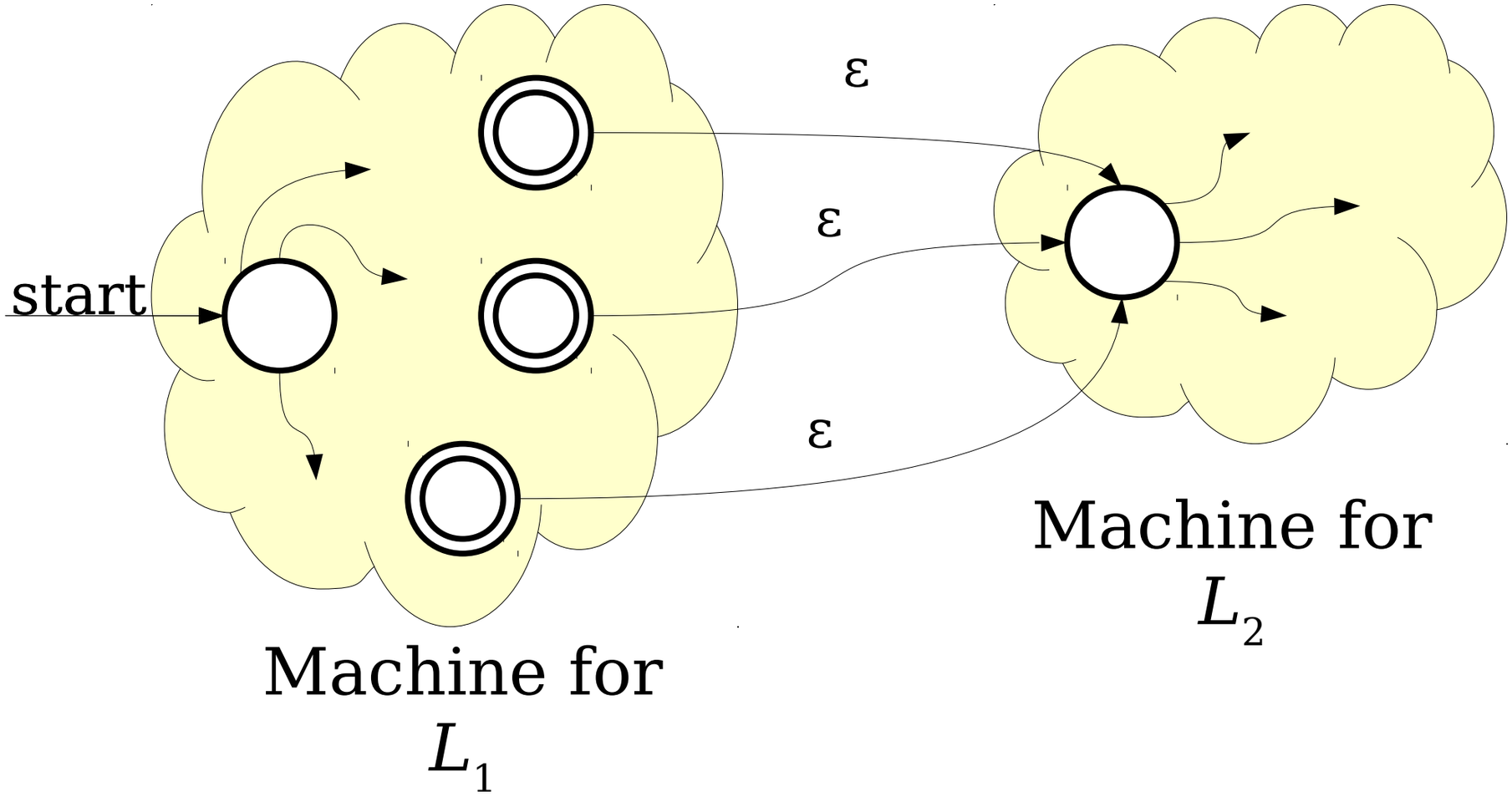


Machine for L_1

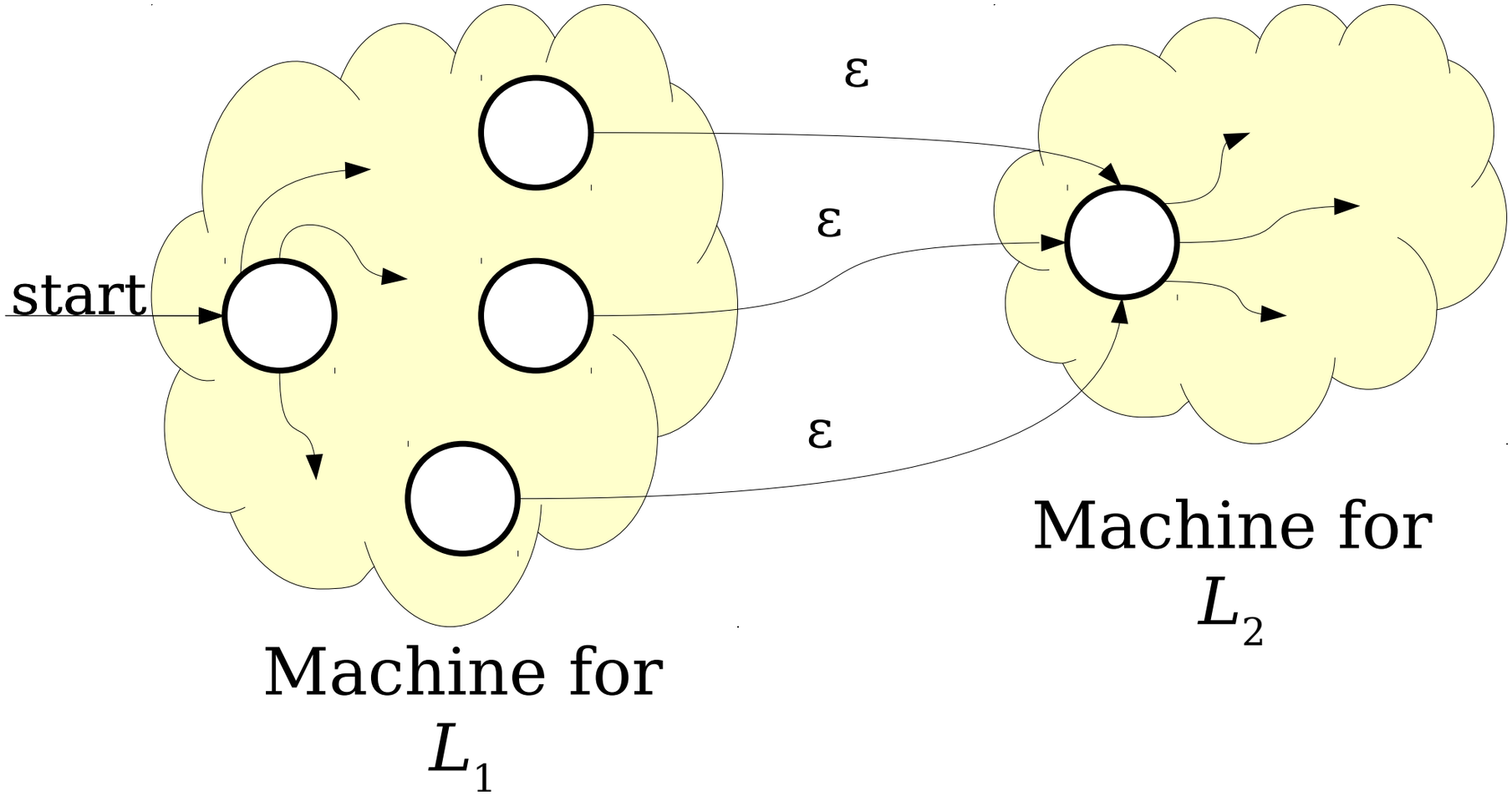


Machine for L_2

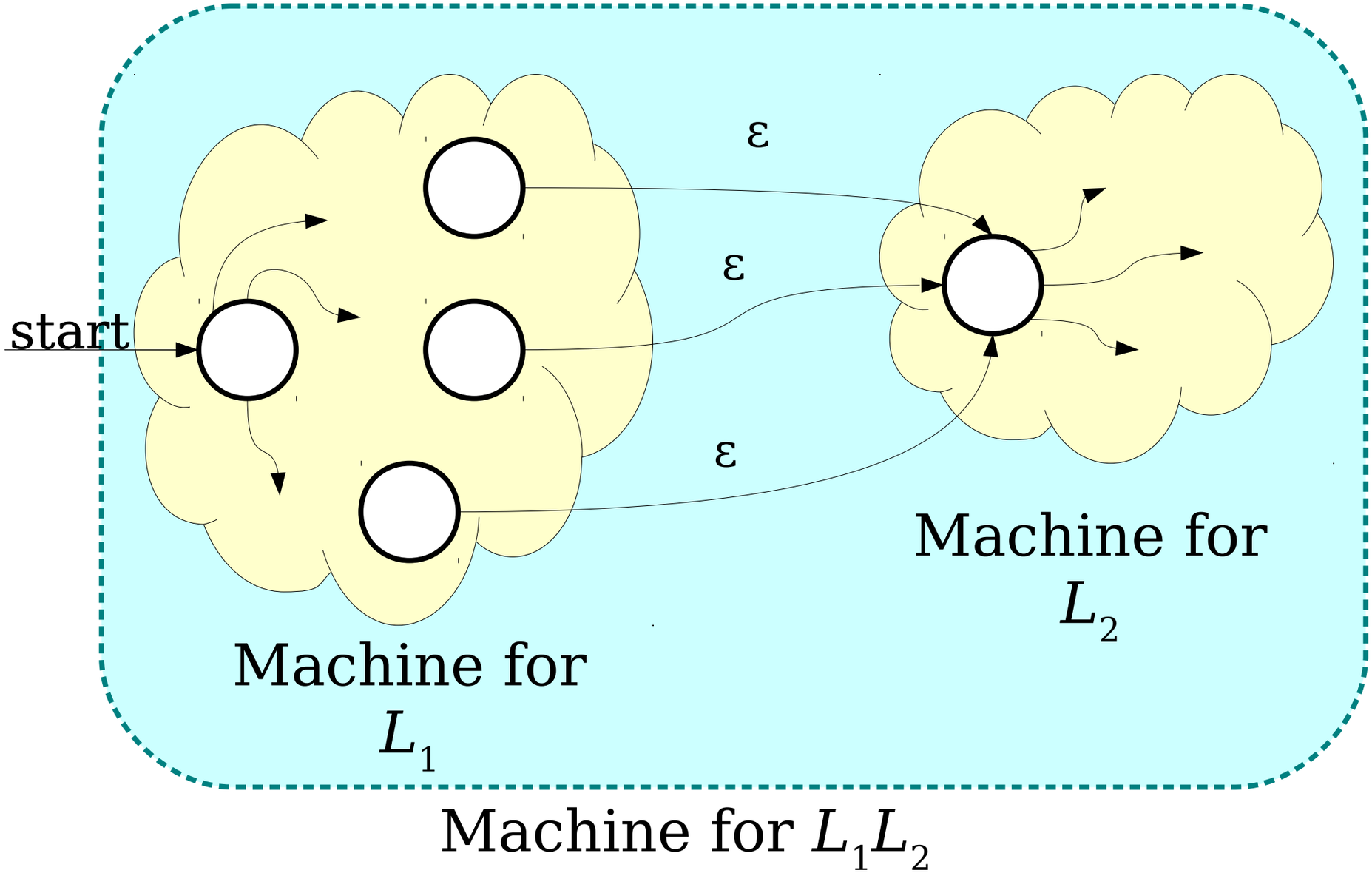
Concatenating Regular Languages



Concatenating Regular Languages



Concatenating Regular Languages



Lots and Lots of Concatenation

- Consider the language L = { **aa**, **b** }
- LL is the set of strings formed by concatenating pairs of strings in L.

{ **aaaa**, **aab**, **baa**, **bb** }

- LLL is the set of strings formed by concatenating triples of strings in L.

{ **aaaaaa**, **aaaab**, **aabaa**, **aabb**, **baaaa**, **baab**, **bbaa**, **bbb**}

- $LLLL$ is the set of strings formed by concatenating quadruples of strings in L.

{ **aaaaaaaa**, **aaaaaab**, **aaaabaa**, **aaaabb**, **aabaaaa**,
aabaab, **aabbaa**, **aabbb**, **baaaaaa**, **baaaab**, **baabaa**,
baabb, **bbaaaa**, **bbaab**, **bbbaa**, **bbbb**}

Language Exponentiation

- We can define what it means to “exponentiate” a language as follows:
- $L^0 = \{\varepsilon\}$
  - Intuition: The only string you can form by gluing no strings together is the empty string.
  - Notice that $\{\varepsilon\} \neq \varnothing$. Can you explain why?
- $L^{n+1} = LL^n$
  - Idea: Concatenating $(n+1)$ strings together works by concatenating n strings, then concatenating one more.
- ***Question to ponder:*** Why define $L^0 = \{\varepsilon\}$?
- ***Question to ponder:*** What is $\varnothing^0$?

The Kleene Star

The Kleene Closure

- An important operation on languages is the ***Kleene Closure***, which is defined as

$$L^* = \{ w \in \Sigma^* \mid \exists n \in \mathbb{N}.\ w \in L^n \}$$

- Mathematically:

$$w \in L^* \quad \leftrightarrow \quad \exists n \in \mathbb{N}.\ w \in L^n$$

- Intuitively, L^* is the language all possible ways of concatenating zero or more strings in L together, possibly with repetition.
- ***Question to ponder:*** What is $\emptyset^*$?

The Kleene Closure

If L = { **a**, **bb** }, then L^* = {

ε,

a, **bb**,

aa, **abb**, **bba**, **bbbb**,

aaa, **aabb**, **abba**, **abbbb**, **bbaa**, **bbabb**, **bbbba**, **bbbbbb**,

...

}

Think of L^* as the set of strings you can make if you have a collection of stamps – one for each string in L – and you form every possible string that can be made from those stamps.

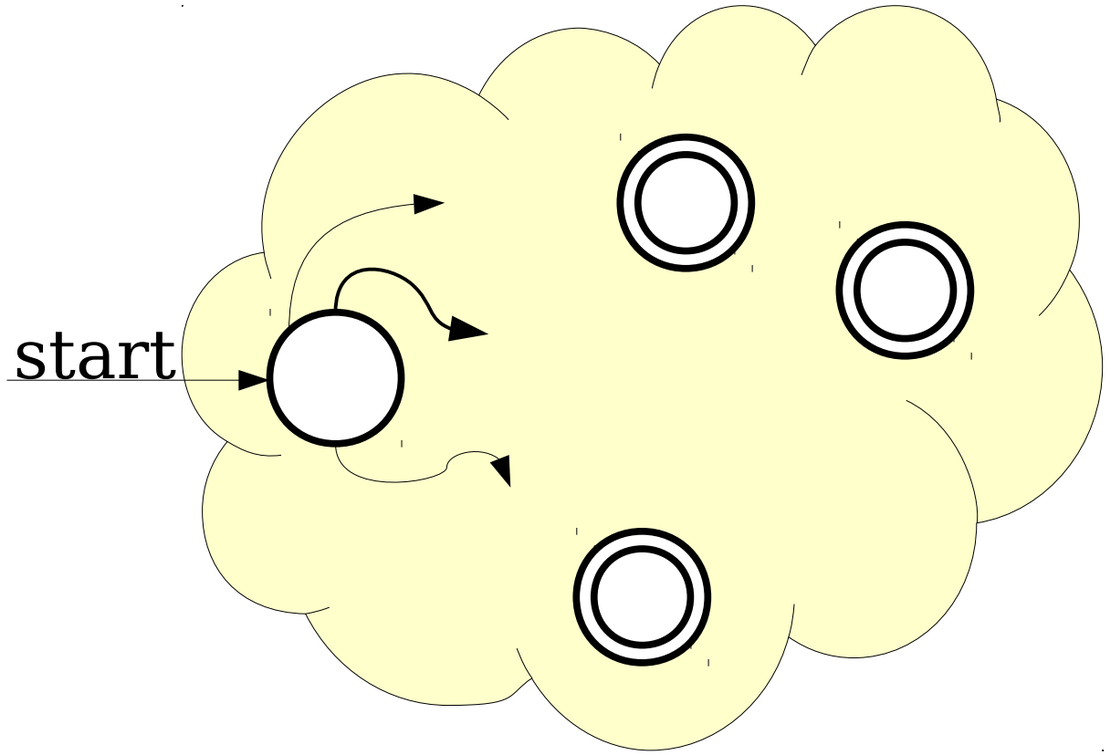
Reasoning about Infinity

- If L is regular, is L^* necessarily regular?
- ⚠ **A Bad Line of Reasoning:** ⚠
  - $L^0 = \{ \varepsilon \}$ is regular.
  - $L^1 = L$ is regular.
  - $L^2 = LL$ is regular
  - $L^3 = L(LL)$ is regular
  - …
  - Regular languages are closed under union.
  - So the union of all these languages is regular.

We won't get into the reasons why, but infinity just doesn't work this way where it neatly extends from the combination of a bunch of finite things.

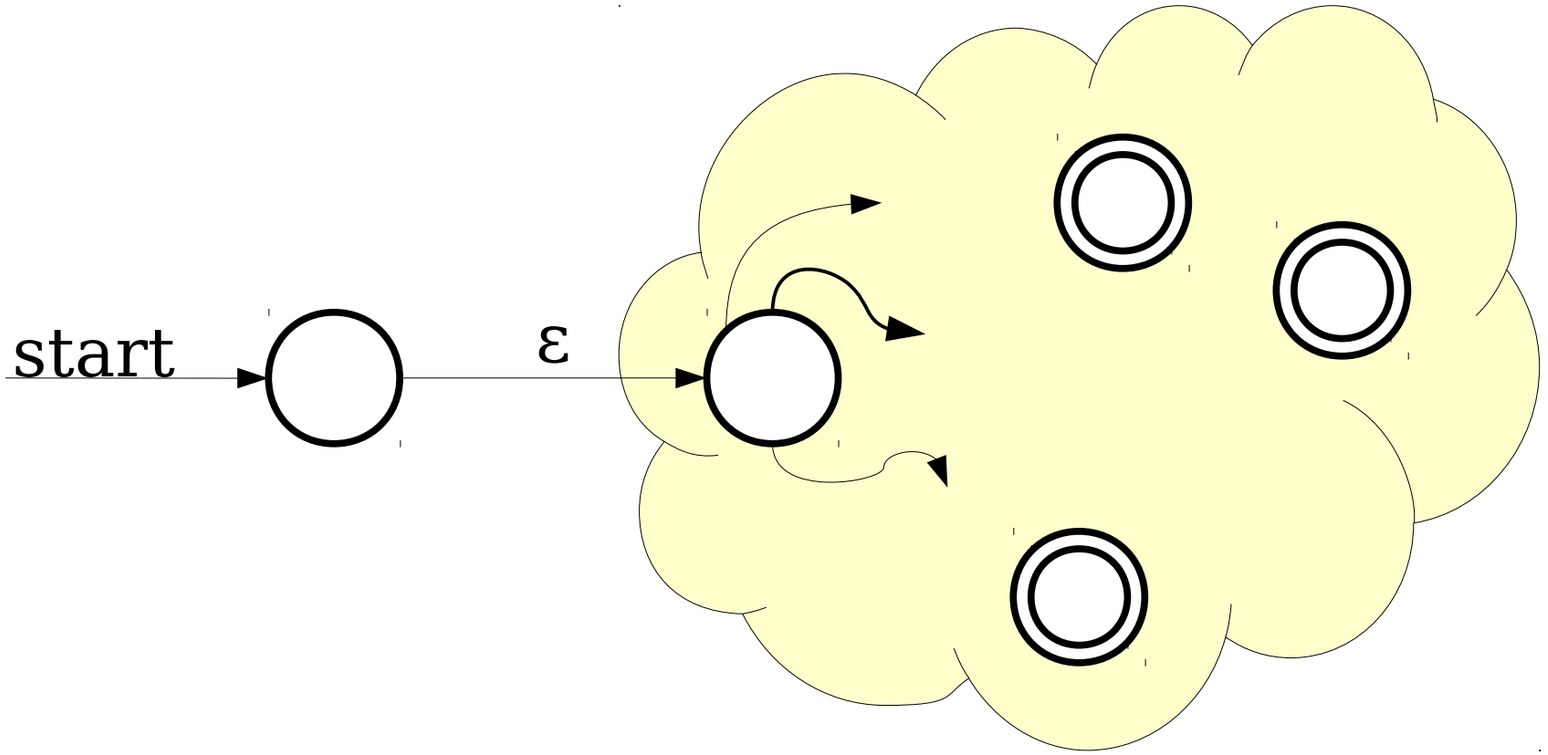
Idea: Can we directly convert an NFA for language L to an NFA for language L^*?

The Kleene Star



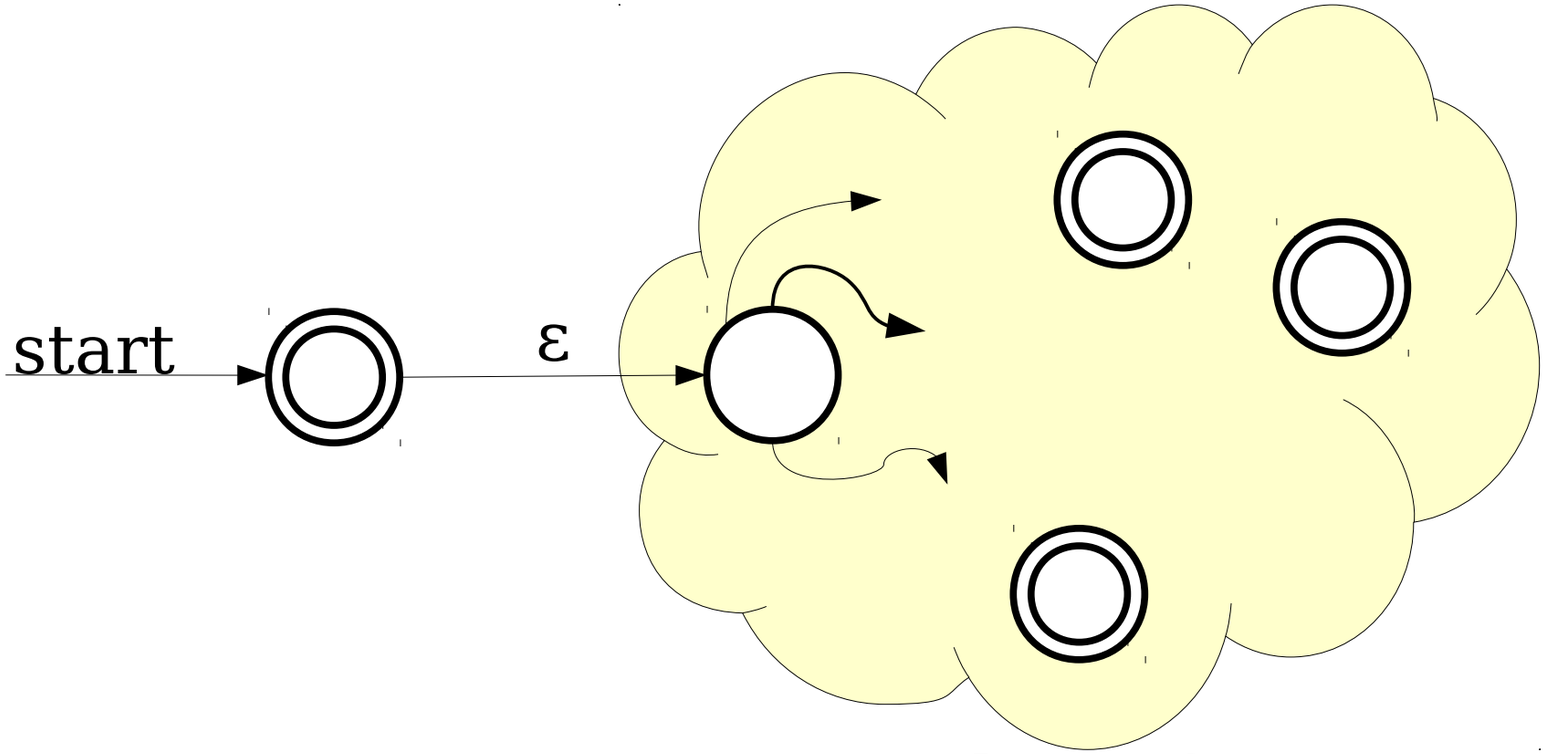
Machine for L

The Kleene Star



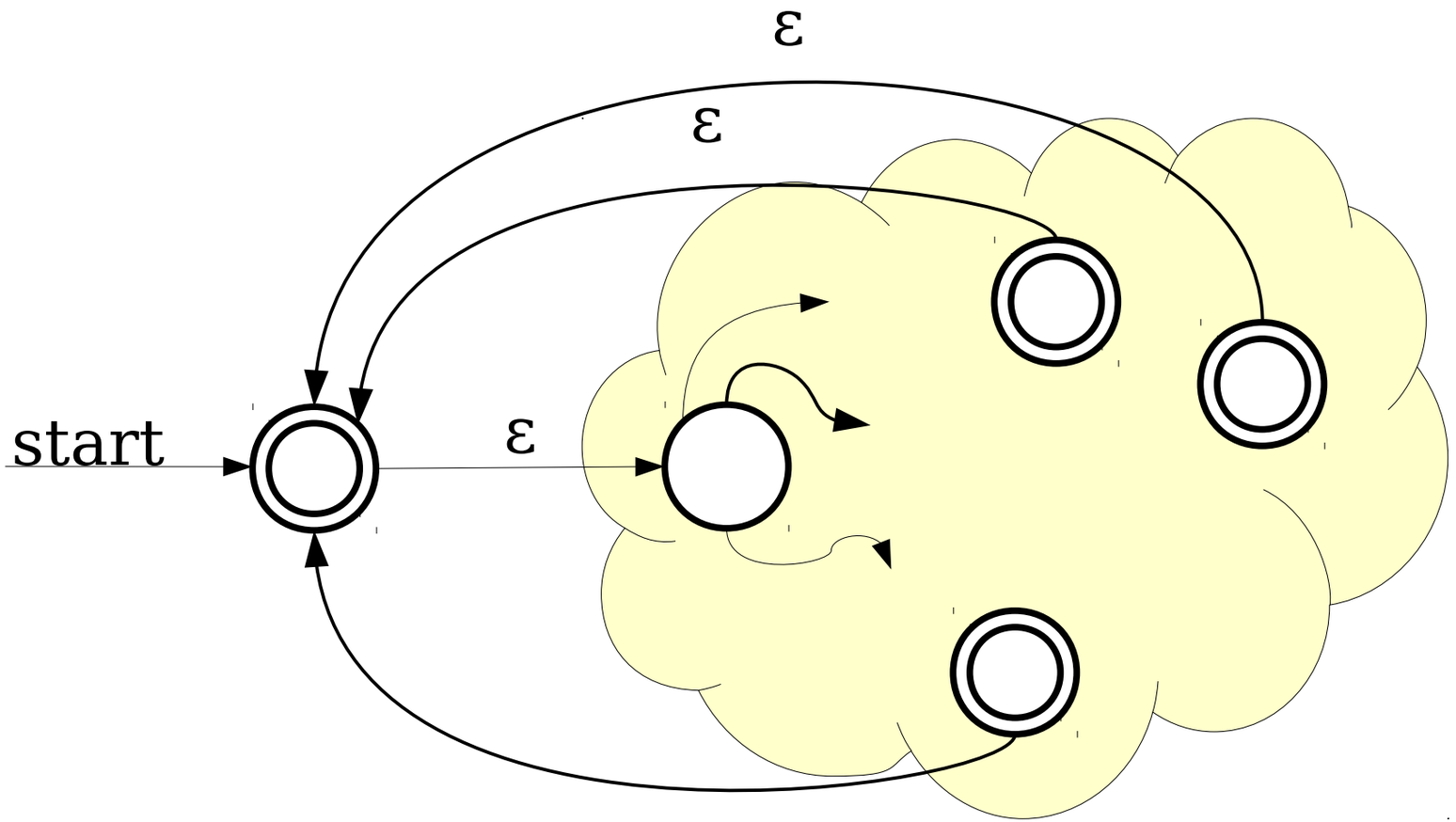
Machine for L

The Kleene Star



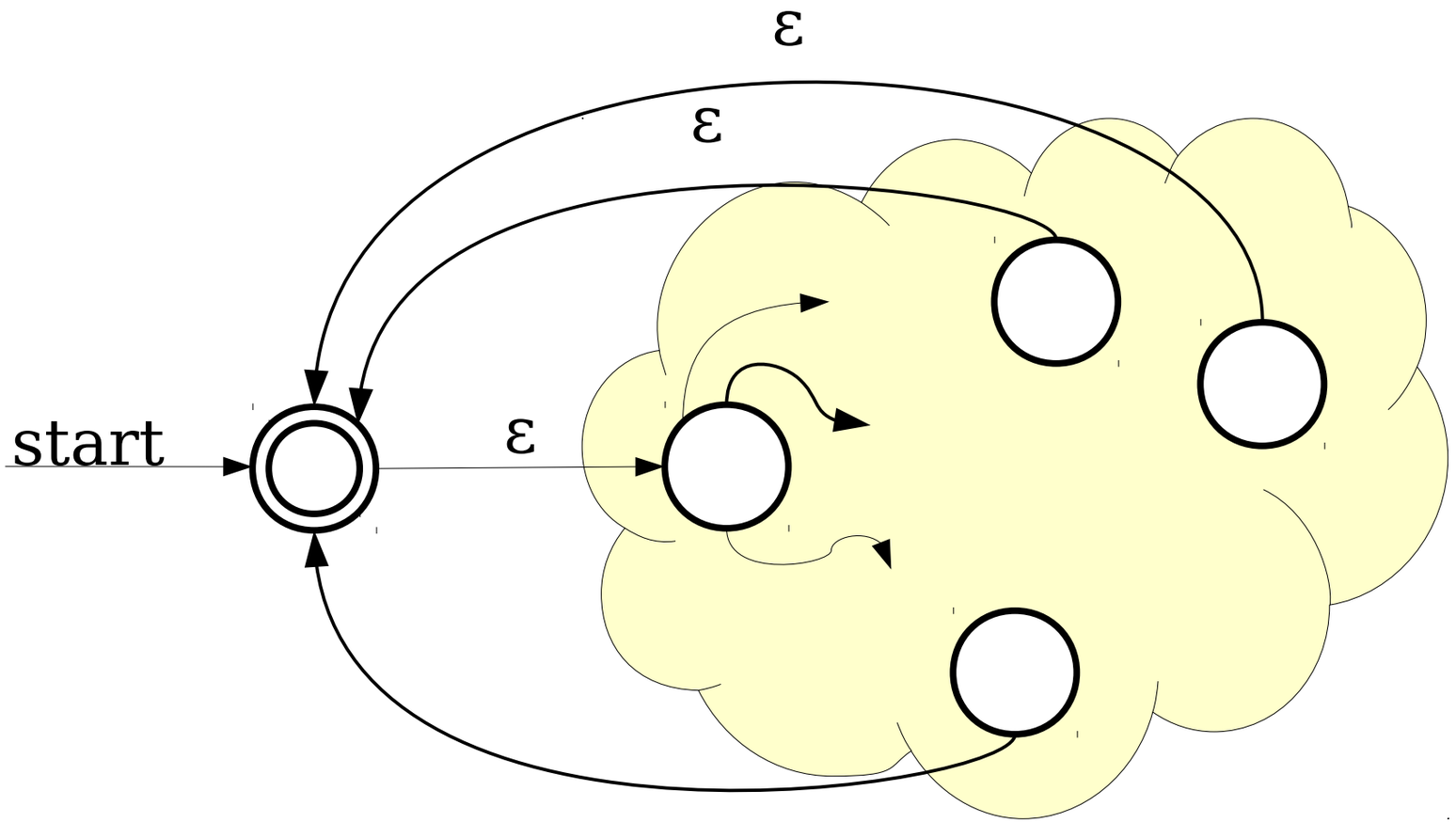
Machine for L

The Kleene Star



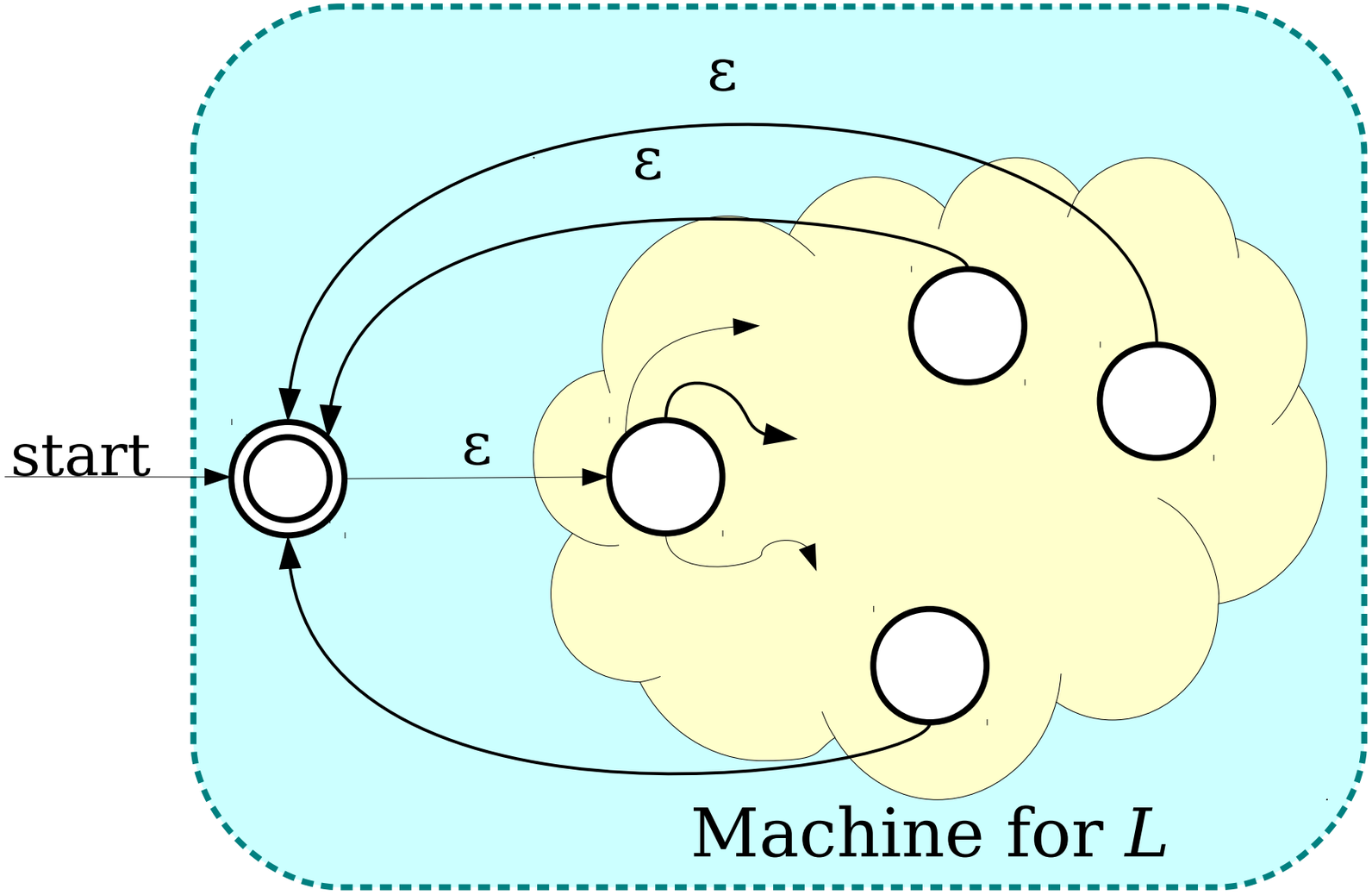
Machine for L

The Kleene Star



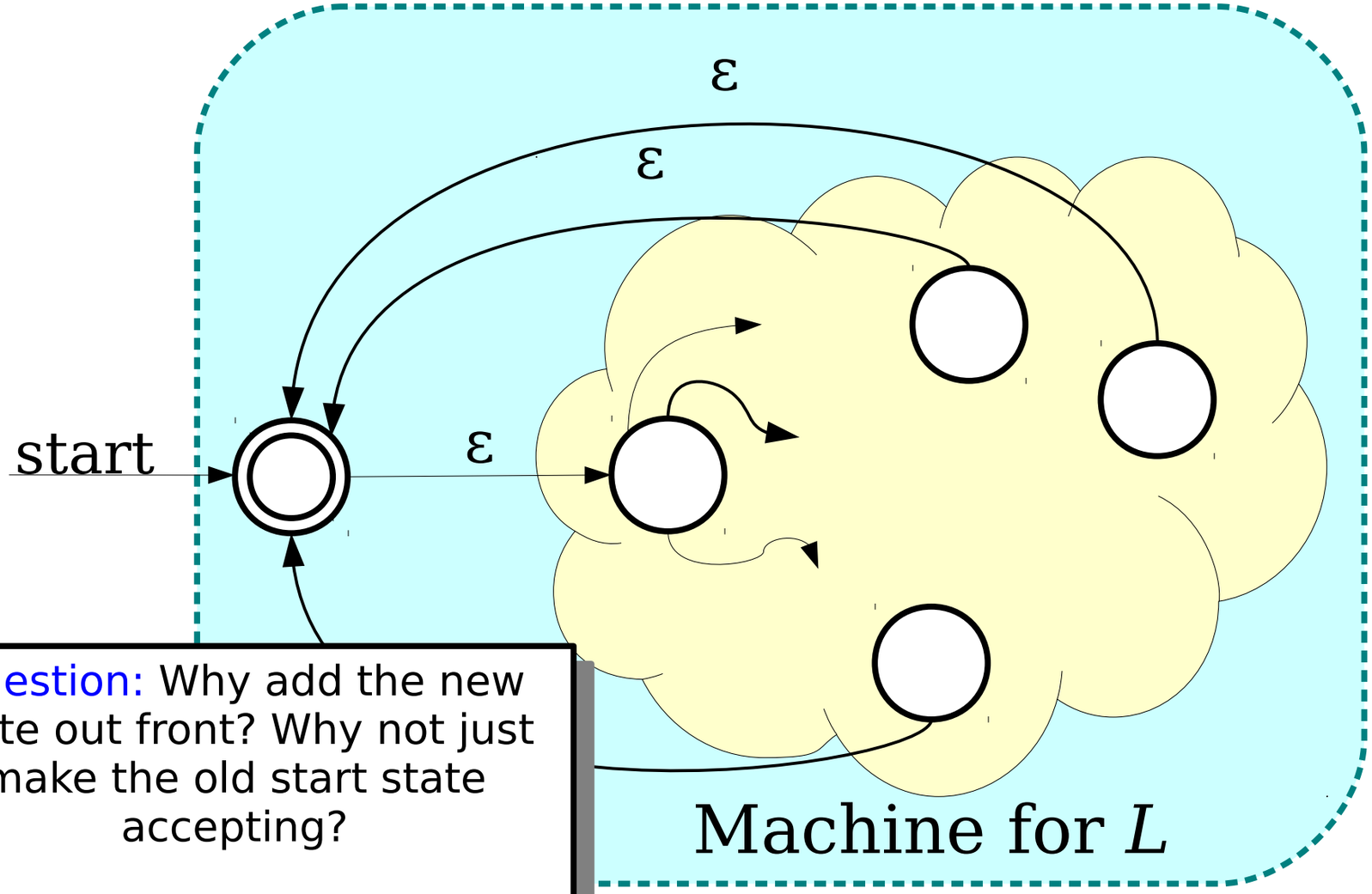
Machine for L

The Kleene Star



Machine for L^*

The Kleene Star



Question: Why add the new state out front? Why not just make the old start state accepting?

Machine for L^*

Closure Properties

- ***Theorem:*** If L_1 and L_2 are regular languages over an alphabet Σ, then so are the following languages:
  - $\overline{L}_1$
  - $L_1 \cup L_2$
  - $L_1 \cap L_2$
  - $L_1 L_2$
  - L_1^*
- These properties are called ***closure properties of the regular languages***.

Next Time

- ***Regular Expressions***
  - Building languages from the ground up!
- ***Thompson's Algorithm***
  - A UNIX Programmer in Theoryland.
- ***Kleene's Theorem***
  - From machines to programs!